Magnetohydrodynamics (MHD) I

Yong-Su Na

National Fusion Research Center

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1. **Review – Confinement & Single Particle Motion**
2. **Plasmas as Fluids – Fluid Equations**
3. **MHD Equations**
4. **MHD Equilibrium**
   - **Concept of Beta**
   - **Equilibrium in the z-Pinch**
   - **Equilibrium in the Tokamak**

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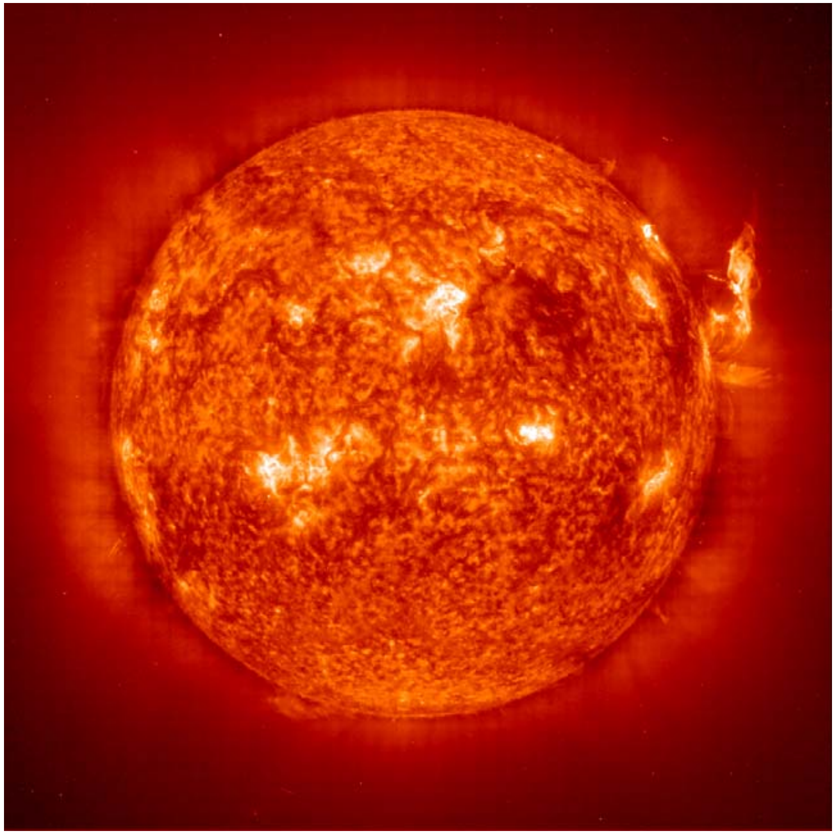
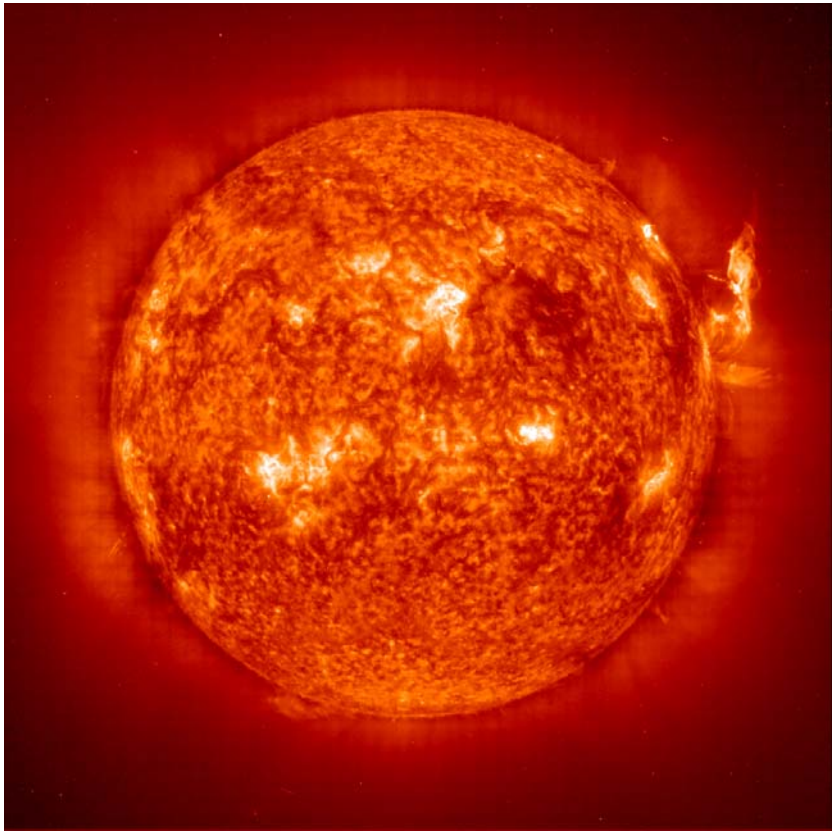
Confinement

Origin of the Star Energy

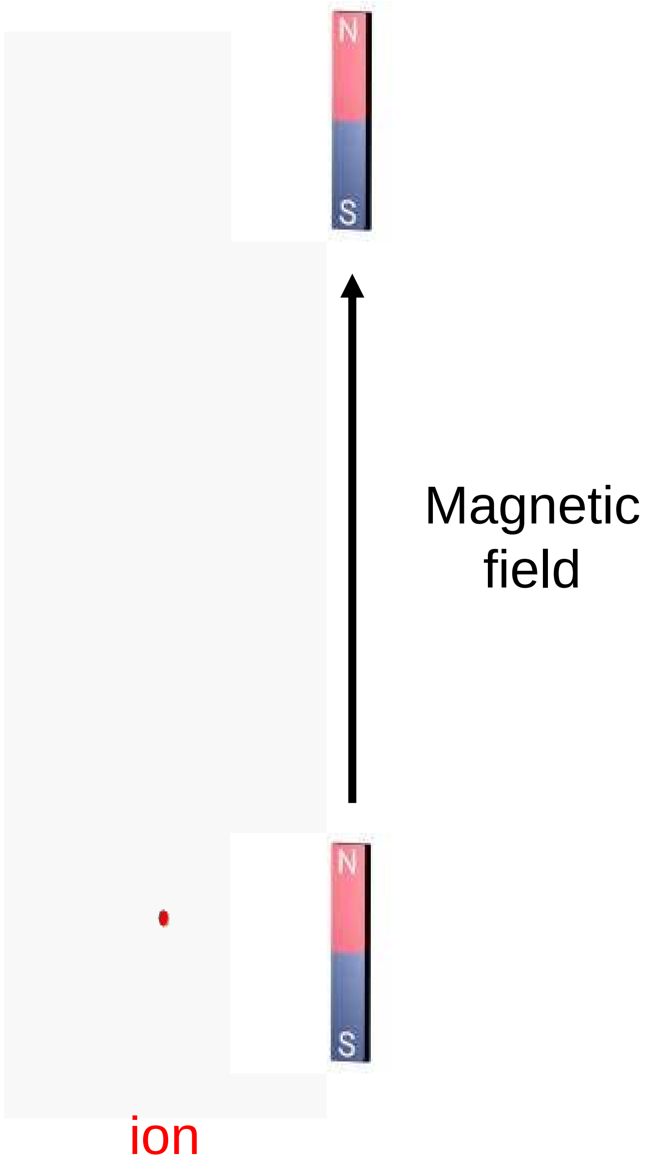


Origin of the Star Energy

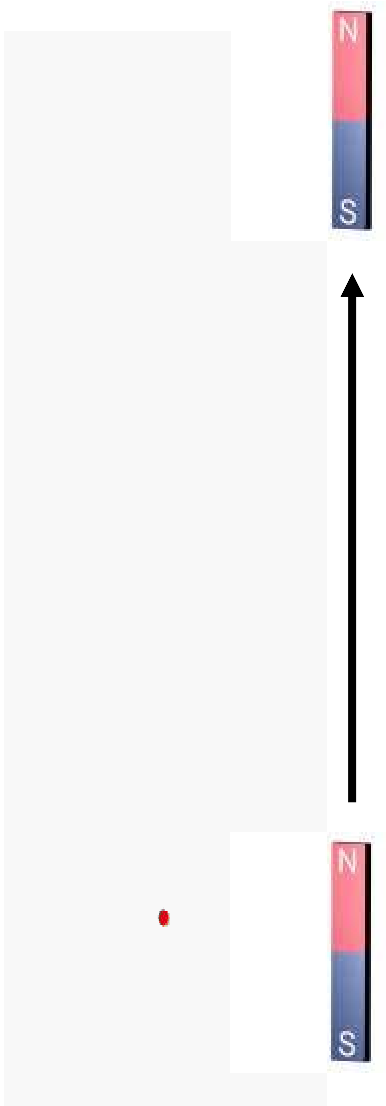
Thermonuclear fusion



How to Confine the Sun on the Earth?



Review:
Single particle motion of the plasma

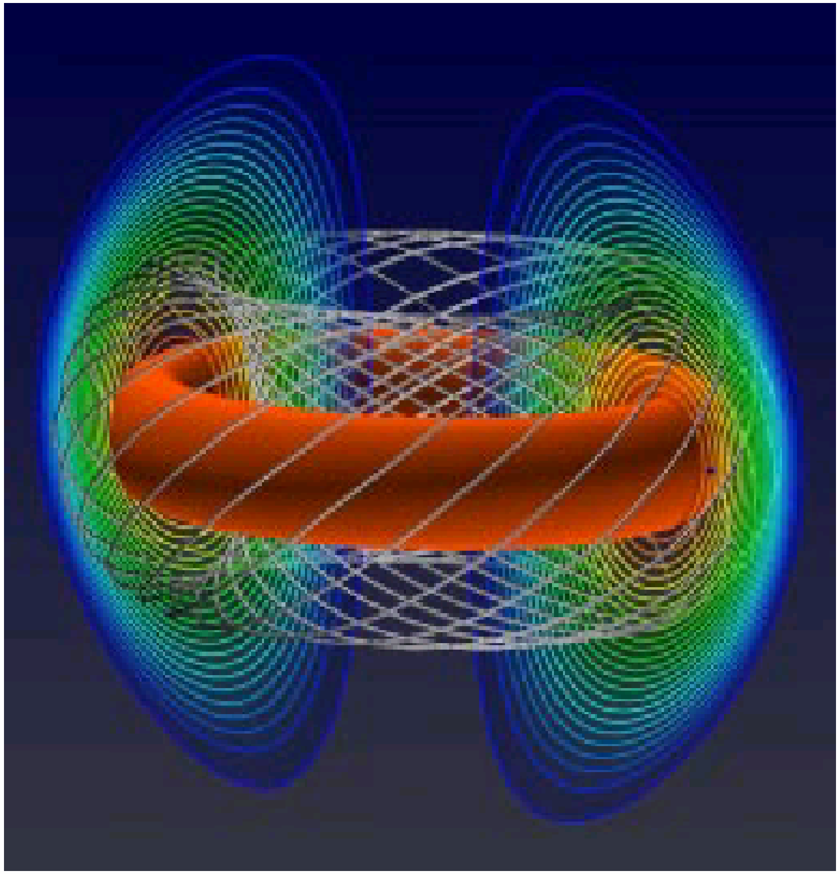


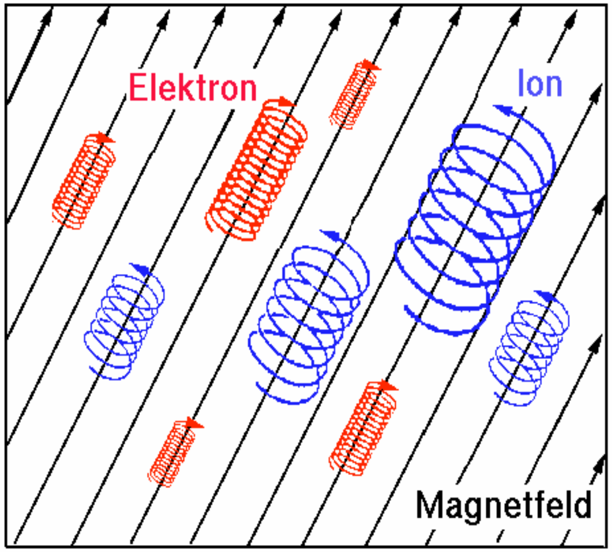
Magnetic field

ion

- free motion along magnetic field lines
- gyration around magnetic field lines

Magnetically Confined Plasmas





Strong magnetic field:
$r_{Lamor} << L$

- Magnetic field lines trace out “magnetic surfaces”, to particles stay on these surfaces.

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Plasmas as Fluids

- The single particle approach gets to be complicated.
- A more statistical approach can be used because we cannot follow each particle separately.
- Now introduce the concept of an **electrically charged current-carrying fluid.**

→ **Magnetohydrodynamic (magnetic fluid dynamic) equations**

Fluid Equations

- Continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (nu) = S$$

S : a volume source rate of particles

- Momentum balance equation

$$mn\frac{du}{dt} = mn[\frac{\partial u}{\partial t} + (u \cdot \nabla)u] = F = nq(E + u \times B)$$

F : force density

When thermal motions are taken into account,

$$mn[\frac{\partial u}{\partial t} + (u \cdot \nabla)u] = nq(E + u \times B) - \nabla \cdot P$$

P : pressure tensor

Fluid Equations

- Momentum balance equation

$$mn[\frac{\partial u}{\partial t} + (u \cdot \nabla)u] = nq(E + u \times B) - \nabla \cdot P$$

$$\rho[\frac{\partial u}{\partial t} + (u \cdot \nabla)u] = -\nabla p + \rho \nu \nabla^2 u$$ Navier-Stokes equation

This momentum density conservation equation for species resembles in parts the one of conventional hydrodynamics, the Navier-Stokes equation. Yet, in a plasma for each species the Lorentz force appears in addition, coupling the plasma motion (via current and charge densities) to Maxwell's equation and also the various components (electrons and ions) among themselves.

- Equation of state

$$p = Cn^{\gamma}$$ $C,\ \gamma$: constant

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Single-fluid MHD

- mass density $\rho = n_i M + n_e m$
- charge density $\sigma = (n_i - n_e)e$
- current density $j = e(n_i u_i - n_e u_e) \approx ne(u_i - u_e)$
- mass velocity $u = (n_i M u_i + n_e m u_e) / \rho$

- Mass and charge continuity equation

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e} u_{i,e}) = 0$$

The individual continuity equations subtracted from one another

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0$$

Multiply by the ion and electron mass and add together

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

Single-fluid MHD

- Momentum balance equation

$$Mn_i[\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla)u_i] = +en_i(E + u_i \times B) - \nabla p_i + R_{ie}$$

$$mn_e[\frac{\partial u_e}{\partial t} + (u_e \cdot \nabla)u_e] = -en_e(E + u_e \times B) - \nabla p_e + R_{ei}$$

$$R_{\alpha\beta} = -m_\alpha n_\alpha < \nu_{\alpha\beta} > (u_\alpha - u_\beta):$$ Momentum of species α transferred by collisions to species β

add together

$$\rho[\frac{\partial u}{\partial t} + u \cdot \nabla u] = \sigma E + j \times B - \nabla p$$

Single-fluid MHD

- generalised Ohm's law

$$mn_e[\frac{\partial u_e}{\partial t}+(u_e\cdot\nabla)u_e]=-en_e(E+u_e\times B)-\nabla p_e+R_{ei}$$

$$R_{ei}=-m_e n_e<\nu_{ei}>(u_e-u_i)$$

Assuming that the electrons are homogeneous and therefore neglecting the electron pressure and velocity gradients along B

$$0=-en_e E_{||}+R_{ei||}\quad j_{||}=-en_e(u_{e||}-u_{i||})$$

$$E_{||}=-\frac{m_e<\nu_{ei}>}{e}(u_{e||}-u_{i||})=\frac{m_e<\nu_{ei}>}{n_e e^2}j_{||}=\eta j_{||}$$

$$R_{ei}=\eta n_e^2 e^2(u_i-u_e)=\eta n_e e j$$

$$E+u_e\times B=\eta j-\nabla p_e/ne\qquad u_e\approx u-j/ne$$

$$E+u\times B=\eta j+(j\times B-\nabla p_e)/ne$$

The set of MHD Equations

KSTAR

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0$$

$$\rho[\frac{\partial u}{\partial t} + u \cdot \nabla u] = \sigma E + j \times B - \nabla p$$

$$E + u \times B = \eta j$$

Simple Ohm‘s law
small Larmor radius approximation

$$\nabla \times B = \mu_0 j$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

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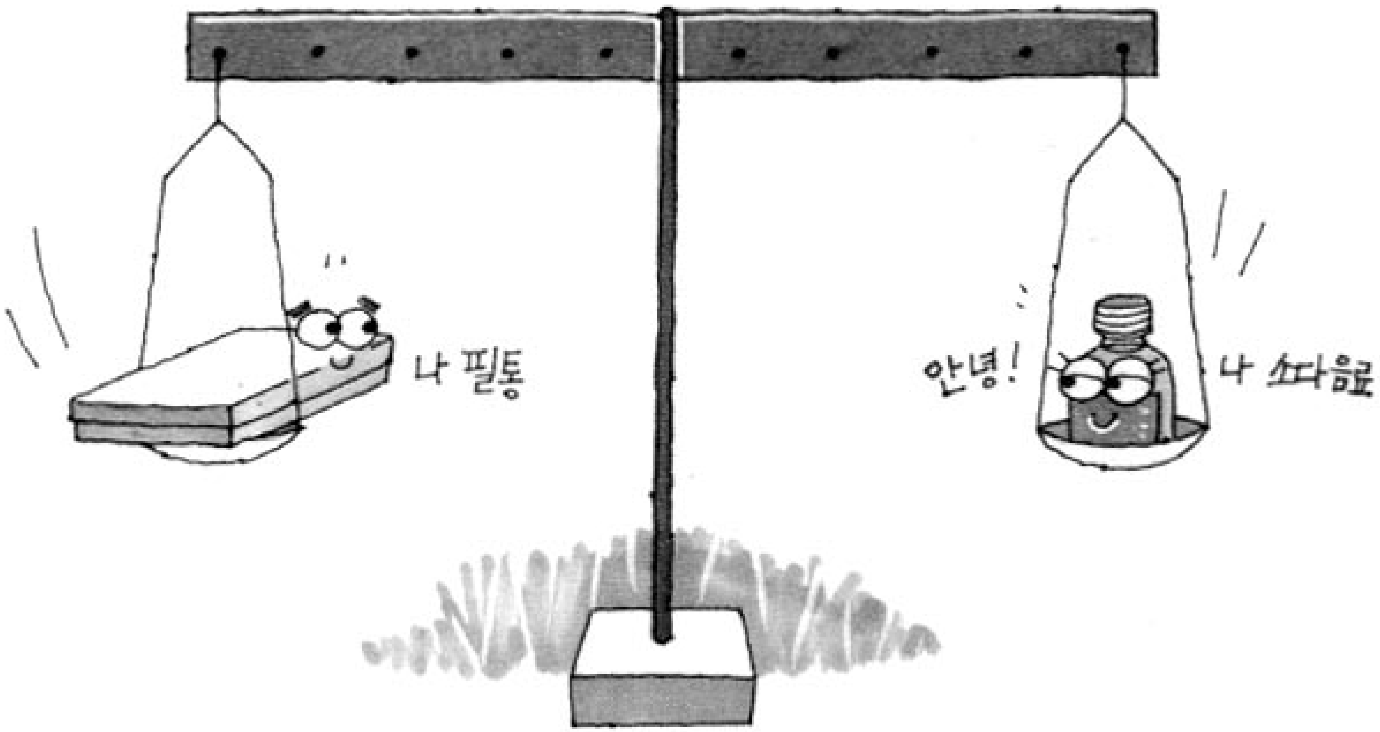
Equilibrium



MHD Equilibrium

- As more and more charged particles are added to a plasma, the currents that flow along the magnetic field can become large enough to modify the externally created magnetic field. The plasma equilibrium must then be determined self-consistently: the presence of the plasma itself modifies the magnetic field configuration.

- For a steady-state solution of the MHD equations for the special case with u=0, E=0, η=0 and isotropic pressure.

MHD Equilibrium

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0$$

$$\rho[\frac{\partial u}{\partial t} + u \cdot \nabla u] = \sigma E + j \times B - \nabla p$$

$$E + u \times B = \eta j$$

$$\nabla \times B = \mu_0 j$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0$$

MHD Equilibrium

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0$$

redundant

$$\rho[\frac{\partial u}{\partial t} + u \cdot \nabla u] = \sigma E + j \times B - \nabla p$$

$$E + u \times B = \eta j$$

$$\nabla \times B = \mu_0 j$$

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$$\nabla \cdot B = 0$$

MHD Equilibrium

$$\nabla p = \vec{j} \times \vec{B}$$ → Force balance

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$ → Ampere's law

$$\nabla \cdot \vec{B} = 0$$ → Closed magnetic field lines

- Two Maxwell's equations (well known)
- One (seemingly) simple force balance:
  kinetic pressure balanced by $\vec{j} \times \vec{B}$ force

MHD Equilibrium

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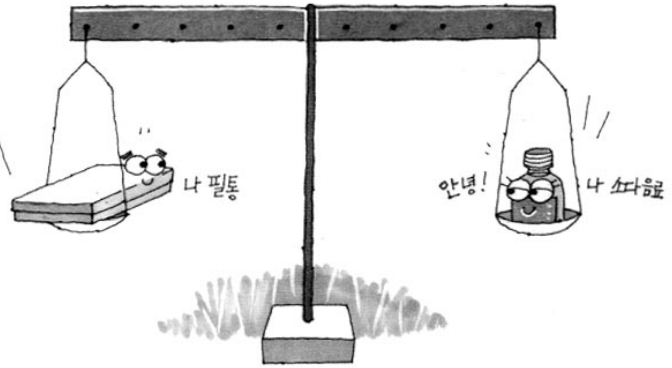
∇p



$\vec{j} \times \vec{B}$

Concept of Beta

$$\nabla p = \vec{j} \times \vec{B}$$ → Force balance

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$$\nabla p = (\nabla \times B) \times B / \mu_0$$
$$= [(B \cdot \nabla) B - \nabla (B^2 / 2)] / \mu_0$$

$$\nabla (p + B^2 / 2\mu_0) = (B \cdot \nabla) B / \mu_0$$

Assuming the field lines are straight and parallel

$$p + B^2 / 2\mu_0 = const.$$

$$\beta = 2\mu_0 p / B^2$$

Concept of Beta

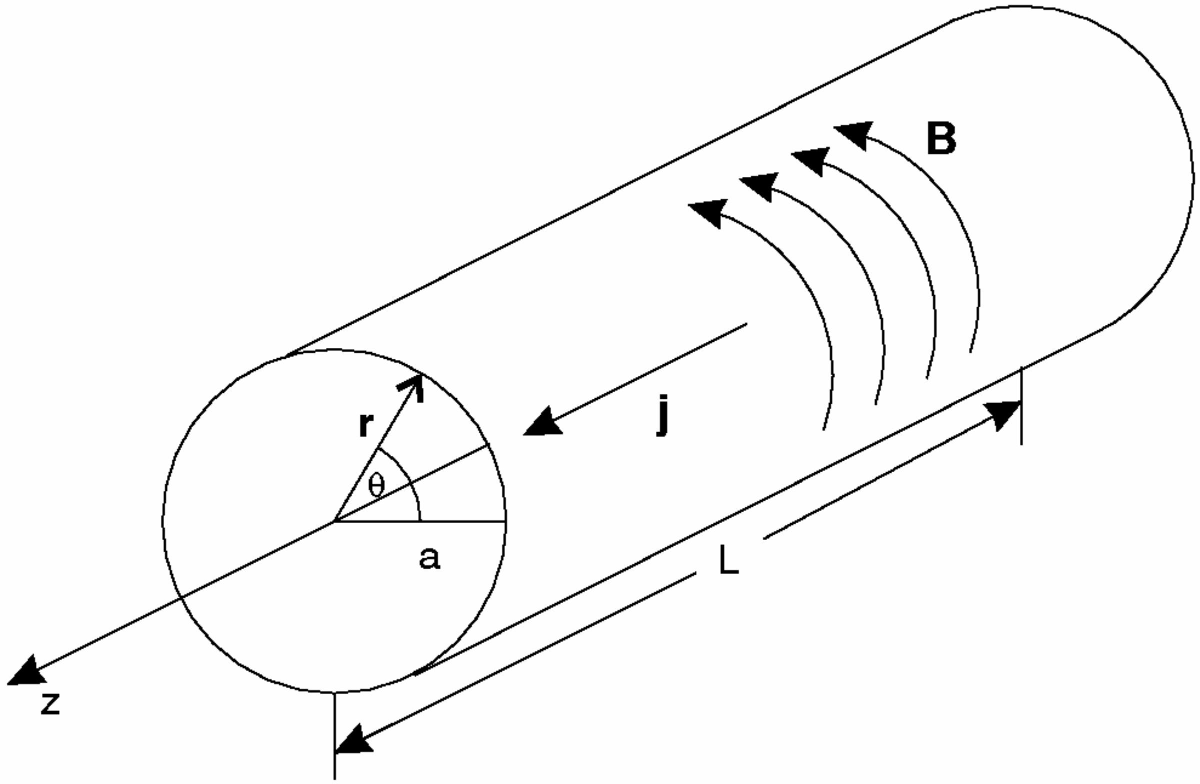
$$\beta = 2\mu_0 p / B^2$$

- The ratio of the plasma pressure to the magnetic field pressure
- A measure of the degree to which the magnetic field is holding a non-uniform plasma in equilibrium

✓ **Main Objectives of Fusion Devices**

- Stable configuration
- High β = $\dfrac{kinetic\ pressure}{magnetic\ pressure} = \dfrac{2\mu_0\langle p\rangle}{B^2} = \dfrac{fusion\ power}{\cos t\ of\ \ decive}$
- High energy confinement time $\tau_E = \dfrac{stored\ energy}{applied\ heating\ power}$

Simplest Case: The Linear Pinch (z-Pinch)



- Cylindrical co-ordinates: $j_z, B_\theta, dp/dr:$

Specify current profile $j_z = j_0 = I_P/(\pi a^2)$ for $r < a$

Magnetic Field Profile in the z-Pinch

Ampere: $$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_0 j$$

Magnetic Field Profile in the z-Pinch

Ampere: $$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_0 j$$

$$\rightarrow B_\theta = \frac{\mu_0}{r}\int_0^r r' j\, dr' = \frac{1}{2\pi}\mu_0 \frac{I(r)}{r} = \frac{\mu_0}{2\pi} I_P \frac{r^2}{a^2}\frac{1}{r}$$

Magnetic Field Profile in the z-Pinch

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$$B_0 = \frac{\mu_0 I_P}{2\pi a^2} r \quad \text{for} \quad r < a$$

$$B_\theta = \frac{\mu_0 I_P}{2\pi r} \quad \text{for} \quad r > a$$

Radial Force Balance and β in the z-Pinch

- With j_z and B_θ : compute force balance:

$$\frac{dp}{dr} = -j_z B_\theta = -\frac{I_P}{\pi a^2}\frac{\mu_0 I_P}{2\pi a^2} r$$

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Boundary condition *p(a)=0*:

$$p(r) = \frac{\mu_0 I_P^2}{2\pi^2 a^4}\left(1 - \left(\frac{r}{a}\right)^2\right)$$

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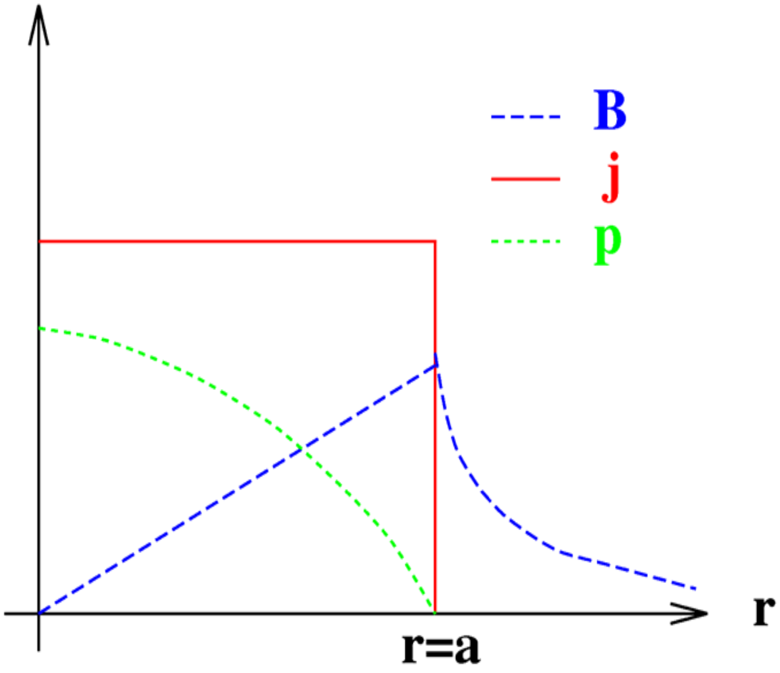

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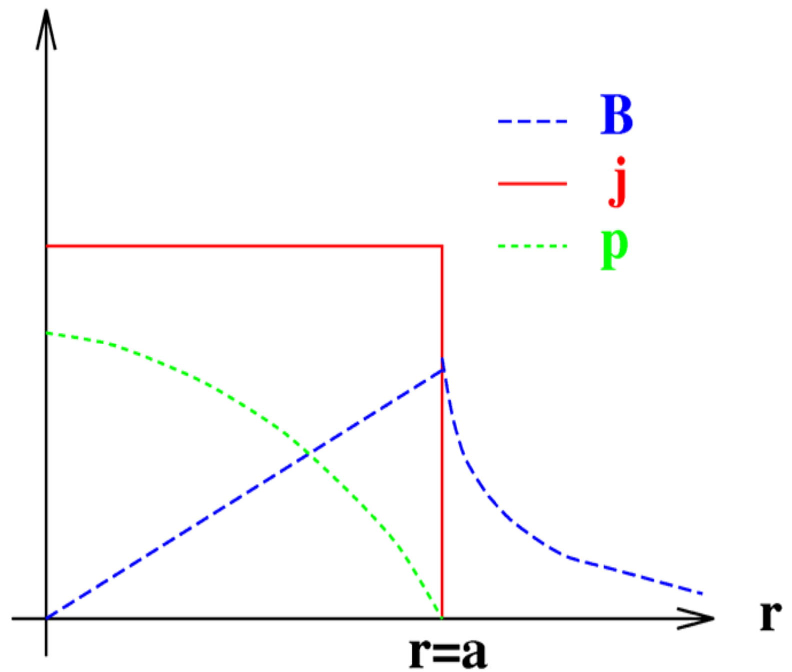

- Calculate $\beta_p = \frac{2\mu_0 < p >}{B_\theta(a)^2}$, we find $\beta_p = 1$

general result for the *z*-pinch, not dependent on profiles.

Equilibrium in the Tokamak

- $\nabla p = \vec{j} \times \vec{B} \Rightarrow \nabla p \cdot \vec{j} = \nabla p \cdot \vec{B} = 0:$

  j and *B* lie in the surfaces *p=const.*

Equilibrium in the Tokamak

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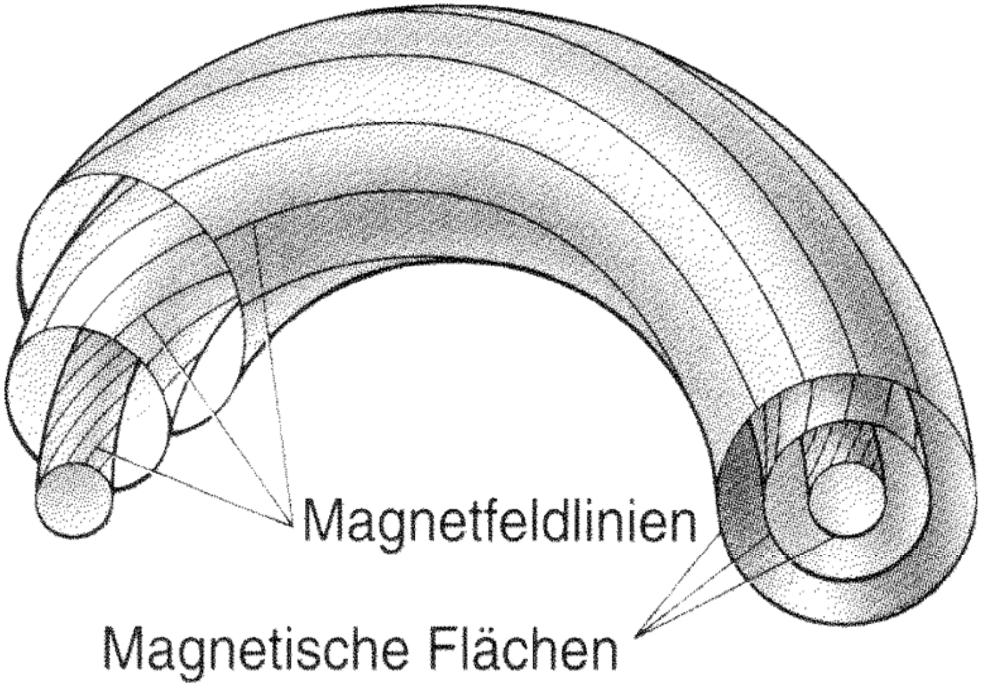
- Flux through any curve on *p=const.* surface has same value:

flux surfaces



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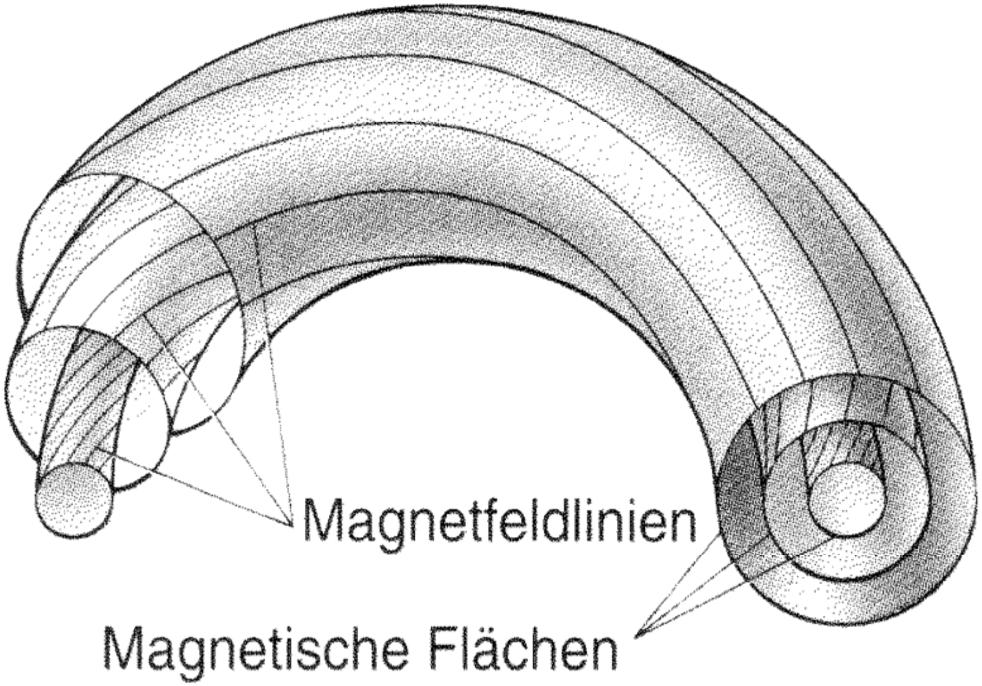

- Pressure is flux surface quantity.
- Tokamak consists of nested flux surfaces set up by *B* field lines

Equilibrium in the Tokamak

- Two sorts of curves on the torus:

winding around poloidally: toroidal fluxes

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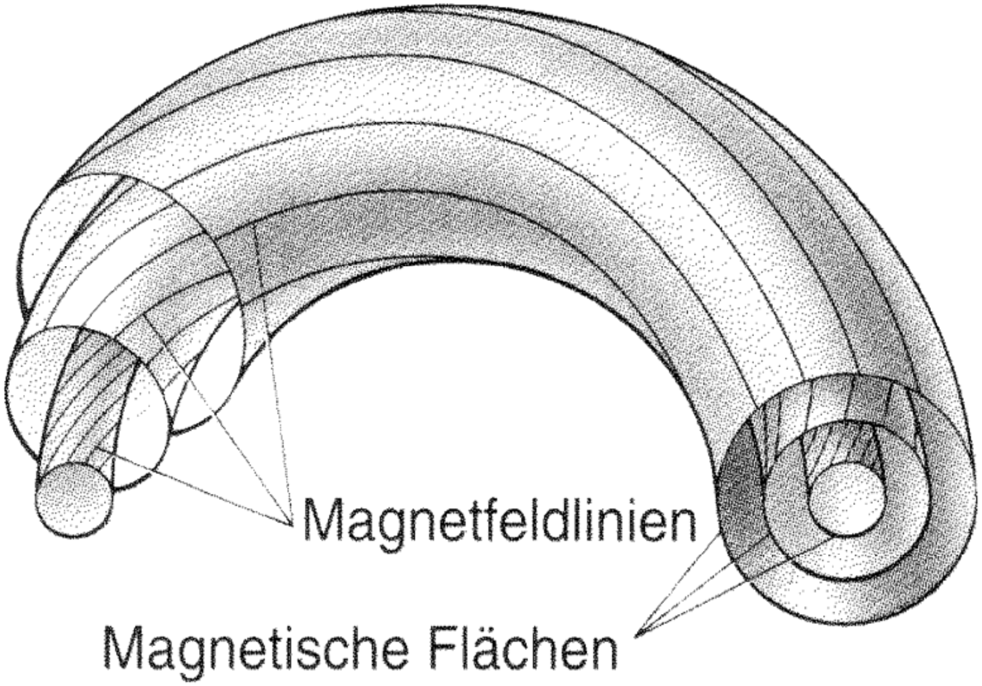


Equilibrium in the Tokamak

- Two sorts of curves on the torus:

  - winding around poloidally: toroidal fluxes

  - winding around toroidally: poloidal fluxes

- Chose the poloidal fluxes ψ (magnetic flux) and I_{pol} (current):

$$B_\phi = \frac{\mu_0 I_{pol}}{2\pi R}$$

$$B_R = -\frac{1}{2\pi R}\frac{\partial \Psi}{\partial Z}$$

$$B_Z = \frac{1}{2\pi R}\frac{\partial \Psi}{\partial R}$$

Grad-Shafranov Equation

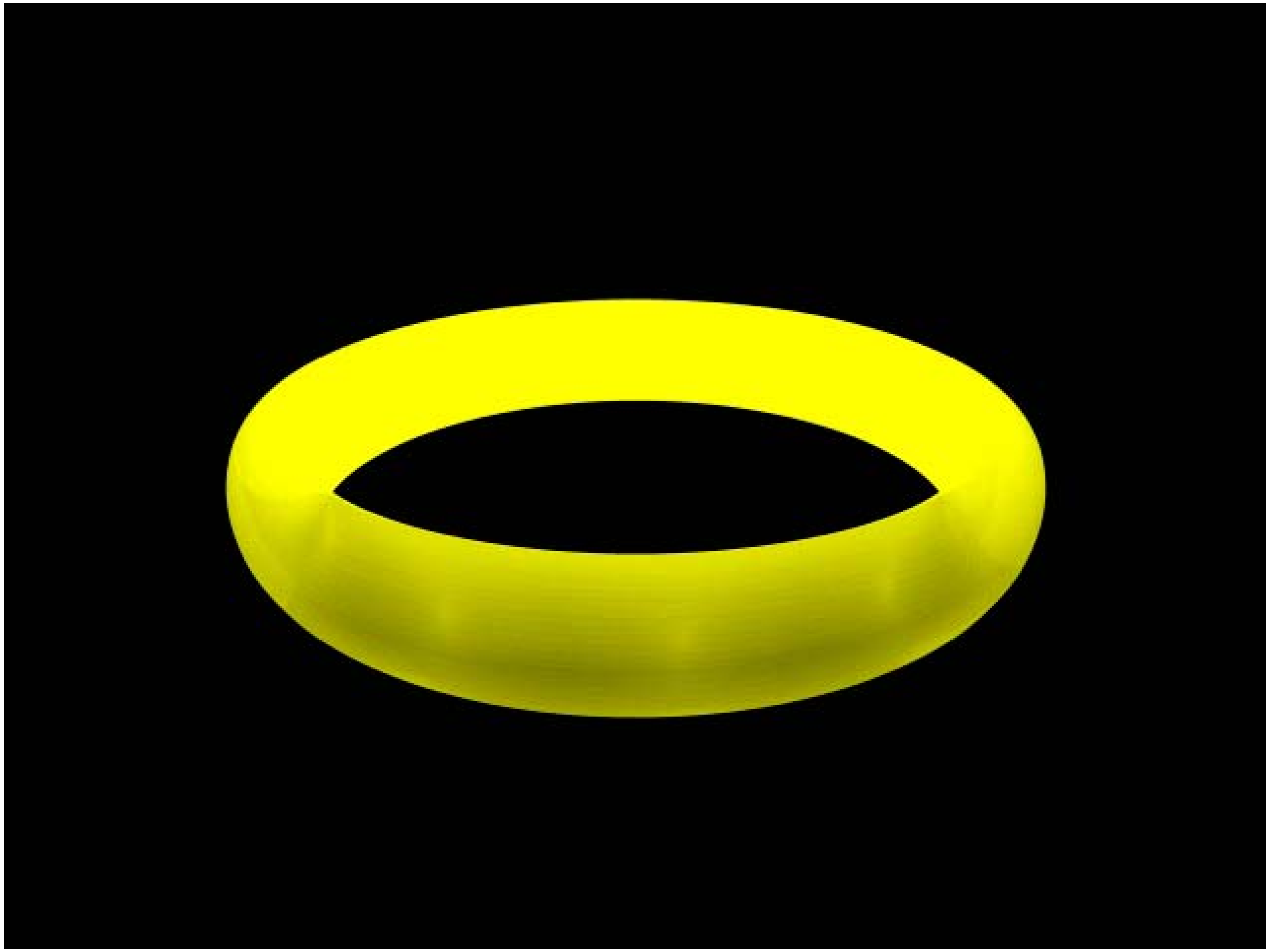
- re-write force balance in terms of fluxes (' denotes *d/d*ψ):

$$-\Delta^*\psi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2} = \mu_0(2\pi R)^2 p' + \mu_0^2 I_{pol} I'_{pol}$$

Grad-Shafranov equation (GS-eqn)

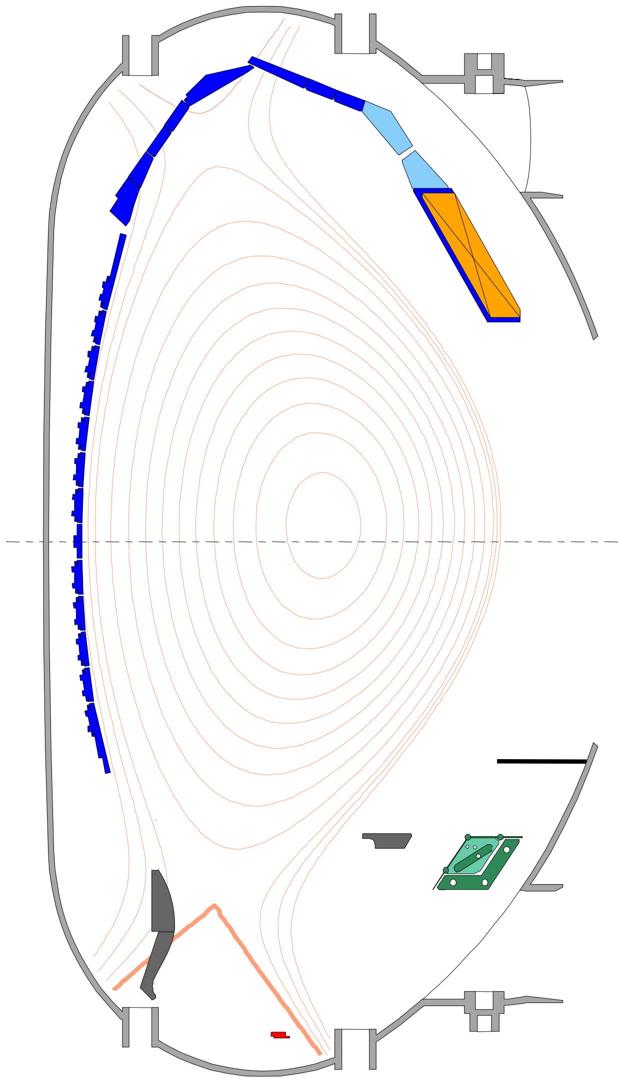
- GS-eqn is nonlinear in ψ. To solve it, one may specify $p(\psi)$ and $I_{pol}(\psi)$.

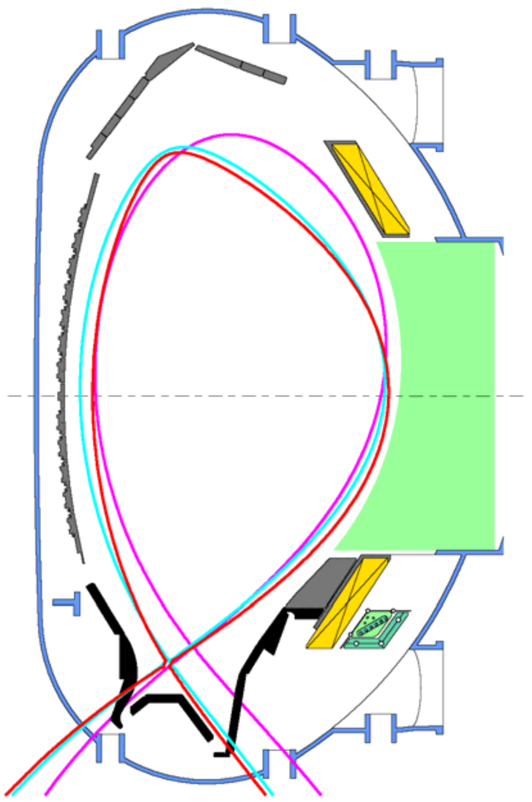
Equilibrium in the Tokamak



Poloidal field coils allow flexible shaping of cross-section

Equilibrium in the Tokamak





- Plasma equilibrium in ASDEX Upgrade

Summary I

1. **What is confinement? Why is single particle motion approach required?**

2. **Fluid description of plasma – Fluid equations**

3. **Single fluid equation – 7 MHD equations**

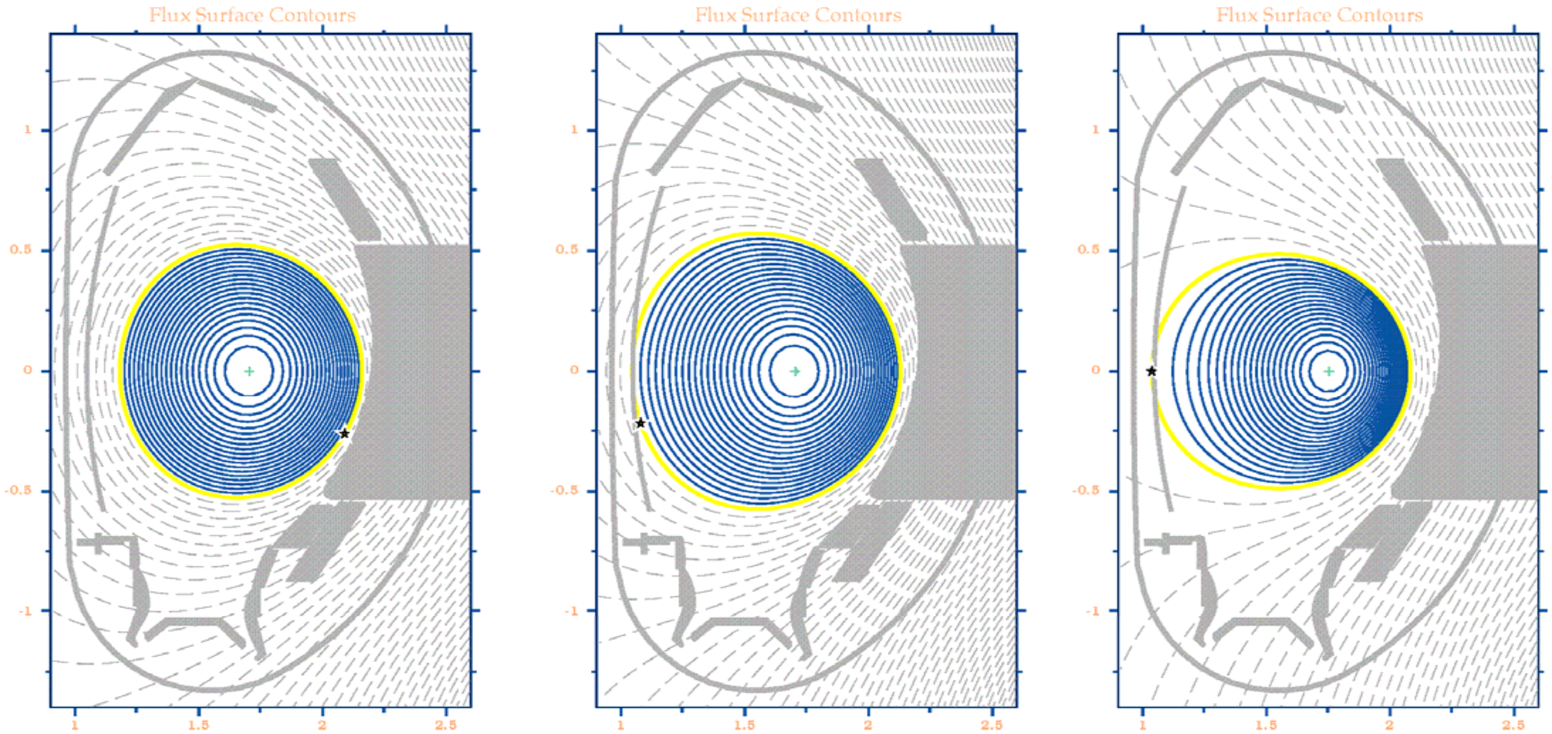
4. **MHD equilibrium**
   - **Concept of beta**
   - **Equilibrium in the z-Pinch**
   - **Equilibrium in the tokamak – GS equation**

The long form of the name MHD means **``magnetic fluid dynamics''**. MHD is a simplified model of a magnetised plasma in which the plasma is treated as a single fluid which can carry an electric current. By single fluid, we mean that there is only one density (the mass density) and temperature, and there is also only one velocity -- you don't have to treat the electrons and ions separately. The basic fluid velocity is the ExB drift, and the difference in the velocities of the electrons and ions appears as the electric current. (Go back to fluid drift motion if any of these terms are unfamiliar.)

The basic physics in the MHD model is the same as for an ordinary fluid, with the addition that the magnetic field can exert a force. The fact that the Lorenz force is opposite for ions and electrons gives rise to a bulk **``magnetic force''** whenever the current flows across magnetic field lines. But since the velocity is the ExB velocity, the magnetic field is advected by the fluid plasma. This couples field and plasma together, giving the MHD system its rich character.

You can think of a tube of magnetic field lines confining a plasma, with the pressure pushing outwards and the magnetic field pushing inwards. This happens because a localised magnetic field always implies the existence of a current, and in this case the current flowing along the tube causes part of the magnetic field to twist around the tube. The sense of it is the ``right-hand rule'' again: with the thumb pointing up and fingers curled back toward the palm, the current flows along the thumb and the magnetic field curls in the direction of the fingers. Now, the force is given by the third direction: J-cross-B, and this points inward towards the axis of the current column, acting to confine the plasma. This is the very basics of how MHD works. The full range of MHD phenomena extends as well to instabilities and turbulence, but this picture of force balance is what is behind the idea of a magnetically confined plasma, one example of which is the **``tokamak''**.

- Poloidal field coils maintain plasma shape and position:



poloidal β = 0.2 poloidal β = 1.2 poloidal β = 3.2

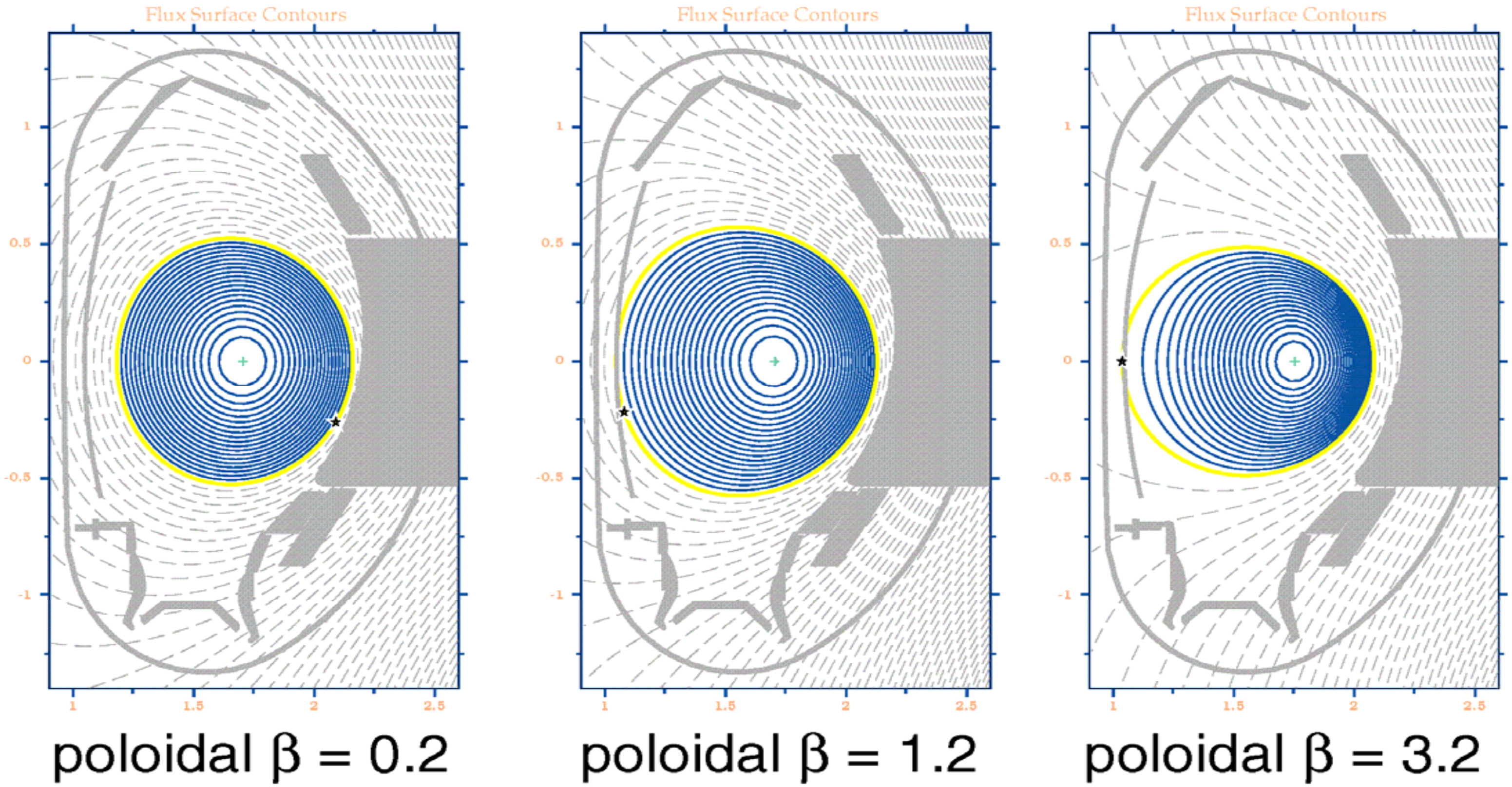
- Vertical field compensates expansion(pressure, current)

  - 'Shafranov-shift' due to larger field on outside

Note: A stellarator produces all fields by external coils

(no net toroidal current and no axisymmetry)

• Poloidal field coils maintain plasma shape and position:



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