6 ECR Heating [or Damping] Rates

6.1 Fund. Harm. Damping Rate - classical approach($\det \overleftrightarrow{M}$)

$$(\overleftrightarrow{M} = \vec{k}\vec{k} + \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon} - k^2 1 = 0)$$

6.1.1 The Dielectric Tensor for $\omega \gg \omega_{pi}$, Ω_i and $\omega \sim |\Omega_e|$

Talking first order in the temperature from the Hot Plasma Dispersion Relation

$$b = \frac{k_\perp^2 T}{m\Omega^2} \ll 1$$

Identities

$$I_n = I_{-n}$$

$$I_n(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+n)!} (\frac{x}{2})^{2s+n}$$

$$I_n(x) = I_{-n}(x)$$

$$\begin{aligned} I_0(b) &= \sum_{s=0}^{\infty} \frac{1}{s!(s)!} (\frac{b}{2})^{2s} \\ &= 1 + \frac{b^2}{4} + \frac{1}{4}\frac{b^4}{16} + \cdots \simeq 1 + \frac{b^2}{4} \end{aligned}$$

$$\begin{aligned} I_1(b) &= \sum_{s=0}^{\infty} \frac{1}{s!(s+1)!} (\frac{b}{2})^{2s} \\ &= \frac{b}{2} + \frac{1}{2}\frac{b^3}{8} + \cdots \simeq \frac{b}{2} \end{aligned}$$

$$\sum_n \Rightarrow n = -1 \ \&\ n = 0 \ \&\ n = 1,$$

$$\sum_s \frac{\omega_{ps}^2}{\omega^2} \simeq \frac{\omega_{pe}^2}{\omega^2} \qquad \text{(only electron, } \omega_{pe} \gg \omega_{pi})$$

$$\epsilon_{xx} = 1 + \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\frac{1-b}{b}[I_1 Z_1 + I_{-1}Z_{-1}] = 1 + \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\frac{1-b}{b}I_1(Z_1 + Z_{-1})$$

$$\epsilon_{yy} = 1 + \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\frac{1-b}{b}I_1(Z_1 + Z_{-1}) - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}2b(1-b)(I_0' - I_0)Z_0$$

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(1-b)[I_0\zeta_0 Z_0' + I_1(\zeta_1 Z_1' + \zeta_{-1}Z_{-1}')]$$

$$\epsilon_{xy} = -\epsilon_{yx} = i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(1-b)[(I_1' - I_1)Z_1 + (I_{-1} - I_{-1}')Z_{-1}]$$

$$= i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(1-b)(I_1' - I_1)(Z_1 - Z_{-1})$$

$$\epsilon_{xz} = \epsilon_{zx} = -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\left(\frac{(1-b)}{\sqrt{2b}}\right)I_1(Z_1' + Z_{-1}')$$

$$\epsilon_{yz} = -\epsilon_{zy} = i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\left(\sqrt{\frac{b}{2}}\right)(1-b)[(I_1' - I_1)(Z_1' - Z_{-1}') + (I_0' - I_0)Z_0']$$

Since

$$I_0(b) = 1 + \frac{b^2}{4} + \cdots \quad \& I_1(b) = \frac{b}{2} + \cdots$$

$$I_0'(b) - I_0(b) = \left(\frac{b}{2} - 1 - \frac{b^2}{4}\right),$$

and the large argument expansion of $Z_n(\zeta)$, we may neglect the last terms in ϵ_{yy} and ϵ_{yz}. Then $\epsilon_{xx} = \epsilon_{yy}$

Note that

$$\zeta_n = \frac{\omega - n\Omega_s}{k_z v_e}$$

For electrons, $n = -1$ is the resonant term.

$$\zeta = \zeta_{-1} = \frac{\omega + \Omega_s}{k_z v_e},$$

$$\frac{\omega - n\Omega_s}{k_z v_e} \gg 1 \quad \text{for } n \neq -1 \ (\Omega_e < 0)$$

$$Z(x) \underset{x \gg 1}{\longrightarrow} i\sqrt{\pi}e^{-x^2} - \frac{1}{x}(1 + \frac{1}{2x^2} + \cdots)$$

: non-resonant term (large argument expansion)

$$
\begin{aligned}
\epsilon_{xx} &= 1+\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\frac{1-b}{b}\cdot\frac{b}{2}(Z_1+Z_{-1}) \\
&= 1+\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z a}(1-b)(Z_1+Z_{-1}) \\
&= 1+\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}Z(\zeta)+\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}\left(-\frac{1}{\zeta_1}\right) \\
&= 1-\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}\cdot\frac{k_z v_e}{\omega+|\Omega_e|}+\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}Z(\zeta) \\
&= 1-\frac{\omega_{pe}^2}{2\omega(\omega+|\Omega_e|)}+\frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}Z(\zeta)=\epsilon_{yy}
\end{aligned}
$$

where $a=v_e\equiv\sqrt{2T_e/m_e}$, $Z(\zeta)=Z_{-1}(\zeta_{-1})$: "resonant term"

$$
|\Omega_e|=\left|\frac{eB}{m_e}\right|
$$

$$
\zeta_{-1}=\zeta=\frac{\omega+\Omega_e}{k_z v_e}=\frac{\omega-|\Omega_e|}{k_z v_e}
$$

$$
\begin{aligned}
\epsilon_{zz} &= 1-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(1-b)[I_0\zeta_0 Z_0'+I_1(\zeta_1 Z_1'+\zeta_{-1}Z_{-1}')] \\
&\simeq 1-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}I_0\zeta_0 Z_0'-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}I_1(\underbrace{\zeta_1 Z_1'}_{\simeq\zeta_1\frac{1}{\zeta_1^2}=\frac{1}{\zeta_1}\ll 1}+\zeta_{-1}Z_{-1}') \\
&\simeq 1-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\cdot 1\cdot\zeta_0\left(\frac{1}{\zeta_0^2}\right)-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\cdot\frac{b}{2}\cdot\zeta Z'(\zeta) \\
&= 1-\frac{\omega_{pe}^2}{\omega^2}-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}b\zeta Z'(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{xy}=-\epsilon_{yx} &= +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(1-b)(I_1'-I_1)(Z_1-Z_{-1}) \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(I_1'-I_1)(Z_1-Z_{-1}) \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\left(\frac{1}{2}-\frac{b}{2}\right)(Z_1-Z_{-1}) \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}(Z_1-Z_{-1}) \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\left(-\frac{1}{\zeta_1}\right)-i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}Z(\zeta) \\
&= -i\frac{\omega_{pe}^2}{2\omega(\omega+|\Omega_e|)}-i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}Z(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{xz} = \epsilon_{zx} &= -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\left(\frac{(1-b)}{\sqrt{2b}}\right) I_1(Z_1' + Z_{-1}') \\
&\simeq -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\frac{(1-b)}{\sqrt{2b}}\frac{b}{2}(Z_1' + Z_{-1}') \\
&\simeq -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\sqrt{\frac{b}{2}}(Z_1' + Z_{-1}') \quad \left(\because Z_1' \sim \frac{1}{\zeta_1^2}\right) \\
&\simeq -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z v_e}\sqrt{\frac{b}{2}}Z'(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{yz} = -\epsilon_{zy} &\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\sqrt{\frac{b}{2}}(1-b)(I_1' - I_1)(Z_1' - Z_{-1}') \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\sqrt{\frac{b}{2}}(I_1' - I_1)(Z_1' - Z_{-1}') \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\sqrt{\frac{b}{2}}\frac{1}{2}(1-b)(Z_1' - Z_{-1}') \\
&\simeq +i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}\sqrt{\frac{b}{2}}Z'(\zeta) \\
&= i\epsilon_{xz}
\end{aligned}
$$

Thus, finally we obtain

$$
\begin{aligned}
\epsilon_{xx} = \epsilon_{yy} &= 1 - \frac{\omega_{pe}^2}{2\omega(\omega + |\Omega_e|)} + \frac{\omega_{pe}^2}{2\omega^2}\frac{\omega}{k_z v_e}Z(\zeta) \\
\epsilon_{zz} &= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}b\zeta Z'(\zeta) \\
\epsilon_{xy} = -\epsilon_{yx} &= -i\frac{\omega_{pe}^2}{2\omega(\omega + |\Omega_e|)} - i\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z a}Z(\zeta) \\
\epsilon_{xz} = \epsilon_{zx} &= -\frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{k_z v_e}\sqrt{\frac{b}{2}}Z'(\zeta) \\
\epsilon_{yz} = -\epsilon_{zy} &= i\epsilon_{xz}
\end{aligned}
$$

where $v_e = \sqrt{2T_e/m_e}$, $b = b_e = \frac{k_\perp T}{m_e \Omega_e^2}$,
$|\Omega_e| = |\frac{eB}{m_e}|$, $\zeta = \frac{\omega - |\Omega_e|}{k_\parallel v_e}$

6.1.2 Damping Rates near the ECR Region

$$
M = Det \overleftrightarrow{M} = Det(\vec{k}\vec{k} + \mu_0\epsilon_0\omega^2 \overleftrightarrow{\epsilon} - k^2 \overleftrightarrow{1}) = 0
$$
$$
\Rightarrow det(\vec{N}\vec{N} + \overleftrightarrow{\epsilon} - N^2 \overleftrightarrow{1}) = 0
$$

where $N = kc/\omega$

In tokamak

$$
B_z(x) = B_T R_0(1 - \frac{x}{R})
$$

The argument of the Fried-Conte function

$$\zeta = \frac{\omega - \Omega_e(x)}{k_\parallel v_e} = \frac{\sqrt{m_e}(\omega - \Omega_e(x))}{k_\parallel \sqrt{2T_e}}$$

since $\Omega_e(x) = \frac{eB_z(x)}{m_e} = \frac{eB_T}{m_e}(1 - \frac{x}{R}) = \Omega_e - \Omega_e \frac{x}{R}$
when $\omega = \Omega_e$
$\omega - \Omega_e(x) = \Omega_e \frac{x}{R}$

$$\begin{aligned}\therefore \zeta &= \frac{\Omega_e \frac{x}{R}}{k_\parallel \sqrt{2}} \sqrt{\frac{m_e}{T_e}} = \frac{x\Omega_e}{\sqrt{2}\frac{\omega}{c}N_\parallel} \frac{1}{R} \sqrt{\frac{m_e}{T_e}} \\ &= \frac{x}{\sqrt{2}} \frac{\Omega_e}{\frac{\Omega_e}{c}N_\parallel} \frac{1}{R} \sqrt{\frac{m_e}{T_e}} = \frac{x}{\sqrt{2}} \frac{1}{N_\parallel R} \sqrt{\frac{m_e c^2}{T_e}} \\ &= \frac{x}{\sqrt{2}\Delta}\end{aligned}$$

where $\Delta = RN_\parallel \sqrt{T_e/m_e c^2}$, c is the speed of light.

"the half width of resonance zone"

Let $\sigma = \frac{\omega}{k_\parallel v_e} = \frac{\omega}{\frac{N_\parallel \omega}{c}\sqrt{\frac{2T_e}{m_e}}} = \frac{1}{\sqrt{2}} \frac{1}{N_\parallel \sqrt{\frac{T_e}{m_e c^2}}} = \frac{R}{\sqrt{2}\Delta}$

Now the dielectric tensor components can be written as

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \frac{{\omega_{pe}}^2}{2\omega(\omega + \Omega_e)} + \frac{{\omega_{pe}}^2}{2\omega^2}\sigma Z(\zeta)$$
$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{{\omega_{pe}}^2}{2\omega^2}\sigma b \zeta Z'(\zeta)$$

since $b = \frac{R_\perp^2 T_e}{m_e \Omega_e^2} = \frac{\omega^2 N_\perp^2 T_e}{m_e c^2 \Omega_e^2}$

$$\begin{aligned}\frac{1}{\omega^2}\sigma b &= \frac{1}{\omega^2}\frac{\sigma^2}{\sigma} b = \frac{1}{\omega^2}\frac{1}{\sigma}\frac{1}{2N_\parallel^2 \frac{T_e}{m_e c^2}} \frac{\omega^2 N_\perp^2 T_e}{m_e c^2 \Omega_e^2} \\ &= \frac{1}{2\Omega_e^2}\frac{N_\perp^2}{N_\parallel^2}\frac{1}{\sigma} \\ \Rightarrow \frac{1}{\omega^2}\sigma^2 b &= \frac{1}{2\Omega_e^2}\frac{N_\perp^2}{N_\parallel^2} \\ \Rightarrow \frac{\sigma\sqrt{b}}{\omega} &= \frac{1}{\sqrt{2}}\frac{1}{\Omega_e}\frac{N_\perp}{N_\parallel}\end{aligned}$$

$$\therefore \epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2}{4\Omega_e^2}\frac{N_\perp^2}{N_\parallel^2}\frac{\zeta}{\sigma}Z'(\zeta)$$

$$\epsilon_{xy} = -\epsilon_{yx} = -i\frac{\omega_{pe}^2}{2\omega(\omega+\Omega_e)} - i\frac{\omega_{pe}^2}{2\omega^2}\sigma Z(\zeta)$$

$$\epsilon_{xz} = \epsilon_{zx} = -\frac{\omega_{pe}^2}{2\omega^2}\sigma\sqrt{\frac{b}{2}}Z'(\zeta) = -\frac{\omega_{pe}^2}{2\omega}\frac{\sigma\sqrt{b}}{\omega}\frac{1}{\sqrt{2}}Z'(\zeta)$$

$$= -\frac{\omega_{pe}^2}{2\omega}\frac{1}{\sqrt{2}}\frac{1}{\Omega_e}\frac{N_\perp}{N_\parallel}\frac{1}{\sqrt{2}}Z'(\zeta) = -\frac{\omega_{pe}^2}{4\omega\Omega_e}\frac{N_\perp}{N_\parallel}Z'(\zeta)$$

$$= -\frac{\omega_{pe}^2}{4\omega\Omega_e^2}\frac{N_\perp}{N_\parallel}Z'(\zeta)$$

$$\epsilon_{yz} = -\epsilon_{zy} = i\epsilon_{xz}$$

We define $\alpha = \frac{\omega_{pe}^2}{\Omega_e^2}$ and $F = \frac{1}{4\sigma}\frac{\omega_{pe}^2}{\Omega_e^2}\frac{\zeta}{N_\parallel^2}Z'(\zeta)$

$$\epsilon_{xx} = 1 - \frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z$$

$$\epsilon_{xy} = -i\left(\frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z\right)$$

$$\epsilon_{xz} = -\frac{\alpha}{4N_\parallel}Z'N_\perp$$

$$\epsilon_{zz} = 1 - \alpha - FN_\perp^2$$

Then, the dispersion relation becomes

$$\overleftrightarrow{M} = \begin{vmatrix} N_\perp^2 + \epsilon_{xx} - (N_\perp^2 + N_\parallel^2) & \epsilon_{xy} & N_\perp N_\parallel + \epsilon_{xz} \\ -\epsilon_{xy} & \epsilon_{xx} - (N_\perp^2 + N_\parallel^2) & i\epsilon_{xz} \\ N_\perp N_\parallel + \epsilon_{xz} & -i\epsilon_{xz} & N_\parallel^2 + \epsilon_{zz} - (N_\perp^2 + N_\parallel^2) \end{vmatrix} = 0$$

$$\Rightarrow [N_\perp^2 + \epsilon_{xx} - N_\perp^2 - N_\parallel^2][(\epsilon_{xx} - N_\perp^2 - N_\parallel^2)(N_\parallel^2 + \epsilon_{zz} - N_\perp^2 - N_\parallel^2) - \epsilon_{xz}^2]$$
$$-\epsilon_{xy}[-\epsilon_{xy}(N_\parallel^2 + \epsilon_{zz} - N_\perp^2 - N_\parallel^2) - i\epsilon_{xz}(N_\perp N_\parallel + \epsilon_{xz})]$$
$$+(N_\perp N_\parallel + \epsilon_{xz})[i\epsilon_{xy}\epsilon_{xz} - (\epsilon_{xx} - N_\perp^2 - N_\parallel^2)(N_\perp N_\parallel + \epsilon_{xz})] = 0$$

Left-hand side:

$$
\begin{aligned}
& (\epsilon_{xx} - N_\|^2)[(\epsilon_{xx} - N_\perp^2 - N_\|^2)(\epsilon_{zz} - N_\perp^2) - \epsilon_{xz}^2] + \epsilon_{xy}[\epsilon_{xy}(\epsilon_{zz} - N_\perp^2) + i\epsilon_{xz}(N_\perp N_\| + \epsilon_{xz})] \\
& +(N_\perp N_\| + \epsilon_{xz})[i\epsilon_{xy}\epsilon_{xz} - (\epsilon_{xx} - N_\perp^2 - N_\|^2)(N_\perp N_\| + \epsilon_{xz})] \\
= \; & (\epsilon_{xx} - N_\|^2)[N_\perp^4 - (\epsilon_{xx} - N_\|^2 + \epsilon_{zz})N_\perp^2 + (\epsilon_{xx} - N_\|^2)\epsilon_{zz} - \epsilon_{xz}^2] \\
& +\epsilon_{xy}[-\epsilon_{xy}N_\perp^2 + \epsilon_{xy}\epsilon_{zz} + i\epsilon_{xz}N_\perp N_\| + i\epsilon_{xz}^2] \\
& +(N_\perp N_\| + \epsilon_{xz})[i\epsilon_{xy}\epsilon_{xz} + \epsilon_{xz}N_\perp^2 + N_\perp^3 N_\| - (\epsilon_{xx} - N_\|^2)N_\perp N_\| - \epsilon_{xz}(\epsilon_{xx} - N_\|^2)] \\
= \; & (\epsilon_{xx} - N_\|^2)N_\perp^4 - (\epsilon_{xx} - N_\|^2)(\epsilon_{xx} - N_\|^2)N_\perp^2 + (\epsilon_{xx} - n_\|^2)^2\epsilon_{zz} - \epsilon_{xz}^2(\epsilon_{xx} - N_\|^2) \\
& -\epsilon_{xy}^2 N_\perp^2 + \epsilon_{xy}^2\epsilon_{zz} + i\epsilon_{xy}\epsilon_{xz}N_\| N_\perp + i\epsilon_{xy}\epsilon_{xz}^2 + i\epsilon_{xy}\epsilon_{xz}N_\perp N_\| + i\epsilon_{xy}\epsilon_{xz}^2 + \epsilon_{xz}N_\perp^3 N_\| + \epsilon_{xz}^2 N_\perp^2 \\
& +N_\perp^4 N_\|^2 + \epsilon_{xz}N_\perp^3 N_\| - (\epsilon_{xz} + N_\perp N_\|)(\epsilon_{xx} + \epsilon_{xz}^2 N_\perp^2)N_\perp N_\| \\
& -\epsilon_{xz}(\epsilon_{xz} + N_\perp N_\|)(\epsilon_{xx} - N_\|^2) \\
= \; & \epsilon_{xx}N_\perp^4 + (\epsilon_{xx} - N_\|^2)(\epsilon_{zz} - N_\perp^2) - (\epsilon_{xx} - N_\|^2)(\epsilon_{zz}N_\perp^2 + \epsilon_{xz}^2 + N_\perp^2 N_\|^2 + \epsilon_{xz}N_\| N_\perp + \epsilon_{xz}^2 \\
& +\epsilon_{xz}N_\perp N_\|) - \epsilon_{xy}^2(N_\perp - \epsilon_{zz}) + 2i\epsilon_{xy}\epsilon_{xz}N_\perp N_\| + 2i\epsilon_{xy}\epsilon_{xz}^2 + 2\epsilon_{xz}N_\perp^3 N_\| + \epsilon_{xz}^2 N_\perp^2 \\
= \; & \epsilon_{xx}N_\perp^4 + (\epsilon_{xx} - N_\|^2)(\epsilon_{zz} - N_\perp^2) - (\epsilon_{xx} - N_\|^2)(\epsilon_{zz}N_\perp^2 + 2\epsilon_{xz}^2 + 2\epsilon_{xz}N_\| N_\perp + N_\perp^2 N_\|^2) \\
& +\epsilon_{xy}^2(\epsilon_{zz} - N_\perp) + 2i\epsilon_{xy}\epsilon_{xz}N_\perp N_\| + 2i\epsilon_{xy}\epsilon_{xz}^2 + 2\epsilon_{xz}N_\perp^3 N_\| + \epsilon_{xz}^2 N_\perp^2 \\
= \; & \left(1 - \frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z\right) N_\perp^4 + \left(1 - \frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z N_\|^2\right)\left(1 - \alpha - F N_\perp^2 - n\perp^2\right) \\
& -\left(1 - \frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z - N_\|^2\right)\left[(1 - \alpha - F N_\perp^2)N_\perp^2 + \frac{\alpha^2}{8N_\|^2}Z'^2 N_\perp^2 - \frac{\alpha}{2N_\|}Z' N_\| N_\perp^2 + N_\perp N_\|^2\right] \\
& -\frac{\alpha^2}{16}\left(1 + 4\sigma Z + 4\sigma^2 Z^2\right)\left(1 - \alpha - F N_\perp^2 - N_\perp^2\right) - 2\frac{\alpha}{4}(1 + 2\sigma Z)\frac{\alpha}{4N_\|}Z' N_\perp^2 N_\| \\
& +2\frac{\alpha}{4}(1 + 2\sigma Z)\frac{\alpha^2}{16N_\|^2}Z'^2 N_\perp^2 - 2\frac{\alpha}{4N_\|}Z' N_\| N_\perp^4 + \frac{\alpha^2}{16N_\|^2}Z'^2 N_\perp^4
\end{aligned}
$$

(1) $\underline{N_\perp^4 \text{ Coefficient}}$

$$\left[1-\frac{\alpha}{4}+\frac{\alpha}{2}\sigma Z+\left(1-\frac{\alpha}{4}+\frac{\alpha}{2}\sigma Z-N_\parallel^2\right)F-\frac{\alpha}{2}+\frac{\alpha^2}{16N_\parallel^2}Z'^2\right]$$

$$= \sigma\frac{\alpha}{2}Z+1-\frac{\alpha}{4}+\left(1-\frac{\alpha}{4}-N_\parallel^2\right)F+\frac{\alpha}{2}Z\sigma F-\frac{\alpha}{2}Z'+\frac{\alpha^2}{16N_\parallel^2}Z'^2$$

$$\left(\begin{array}{l} \sigma F=\frac{\alpha}{4}\frac{\zeta}{N_\parallel^2}Z' \quad \text{and} \quad Z'=-2(1+\zeta Z)\Rightarrow \zeta Z=-\frac{1}{2}Z'-1 \\ \therefore \frac{\alpha}{2}Z\sigma F=\frac{\alpha}{2}Z\frac{\alpha}{4}\frac{\zeta}{N_\parallel^2}Z' \\ \qquad\qquad =\frac{\alpha^2}{8N_\parallel^2}Z'\left(-\frac{1}{2}Z'-1\right)=-\frac{\alpha^2}{16N_\parallel^2}Z'^2-\frac{\alpha^2 Z'}{8N_\parallel^2} \end{array}\right.$$

$$= \sigma\frac{\alpha}{2}Z+1-\frac{\alpha}{4}+\left(1-\frac{\alpha}{4}-N_\parallel^2\right)F-\frac{\alpha^2}{16N_\parallel^2}Z'^2-\frac{\alpha^2}{8N_\parallel^2}Z'-\frac{\alpha}{2}Z'+\frac{\alpha^2}{16N_\parallel^2}Z'$$

$$= \sigma\frac{\alpha}{2}Z+1-\frac{\alpha}{4}+\left(1-\frac{\alpha}{4}-N_\parallel^2\right)F-\frac{\alpha}{2}\left(1+\frac{\alpha}{4N_\parallel^2}\right)Z'=\mathbf{A}$$

(2) $\underline{N_\perp^2 \text{ Coefficient}}$

$$\begin{aligned} - \quad & \left(1-\frac{\alpha}{4}+\alpha 2\sigma Z-N_\parallel^2\right)^2(F+1)-\left(1-\frac{\alpha}{4}+\frac{\alpha}{2}\sigma Z-N_\parallel^2\right)\left[(1-\alpha)+\frac{\alpha^2}{8N_\parallel^2}Z'2-\frac{\alpha}{2}Z'+N_\parallel^2\right] \\ & +\frac{\alpha^2}{16}[1+4\sigma Z(1+\sigma Z)](F+1)-\frac{\alpha^2}{8}(1+2\sigma Z)Z'+\frac{\alpha^3}{32N_\parallel^2}(1+2\sigma Z)Z'^2 \\ = \quad & \left[-\left(1-N_\parallel^2-\frac{\alpha}{4}\right)-\alpha\sigma Z\left(1-N_\parallel^2-\frac{\alpha}{4}\right)-\frac{\alpha^2}{4}\sigma^2Z^2\right](F+1) \\ & -\left(1-\frac{\alpha}{4}-N_\parallel^2\right)\left[(1-\alpha)+\frac{\alpha^2}{8N_\parallel^2}Z'^2-\frac{\alpha}{2}Z'+N_\parallel^2\right]-\frac{\alpha}{2}\sigma Z\left[(1-\alpha)+\frac{\alpha^2}{8N_\parallel^2}Z'^2-\frac{\alpha}{2}Z'+N_\parallel^2\right] \\ & +\frac{\alpha^2}{16}\left[1+4\sigma Z(1+\sigma Z)\right](F+1)-\frac{\alpha^2}{8}Z'-\frac{\alpha^2}{4}\sigma ZZ'+\frac{\alpha^3}{32N_\parallel^2}Z'^2+\frac{\alpha^3}{16N_\parallel^2}\sigma ZZ' \\ = \quad & -(1-N_\parallel^2)\left(1-N_\parallel^2-\frac{\alpha}{2}\right)F-\frac{\alpha^2}{16}F-\left(1-N_\parallel^2-\frac{\alpha}{4}\right)^2-\alpha\sigma\ ZF\left(1-N_\parallel^2-\frac{\alpha}{4}\right) \\ & -\alpha\sigma Z\left(1-N_\parallel^2-\frac{\alpha}{4}\right)-\frac{\alpha^2}{4}\sigma^2Z^2F-\frac{\alpha^2}{4}\sigma^2Z^2-\left(1-\frac{\alpha}{4}-N_\parallel^2\right)(1-\alpha) \\ & -\frac{\alpha^2}{8N_\parallel^2}\left(1-\frac{\alpha}{4}-N_\parallel^2\right)Z'^2+\frac{\alpha}{2}\left(1-\frac{\alpha}{4}-N_\parallel^2\right)Z'-\left(1-\frac{\alpha}{4}-N_\parallel^2\right)(1-\alpha) \\ & -\frac{\alpha}{2}\sigma Z(1-\alpha)-\frac{\alpha^3}{16N_\parallel^2}\sigma ZZ'^2+\frac{\alpha^2}{4}\sigma ZZ'-\frac{\alpha}{2}N_\parallel^2\sigma Z+\frac{\alpha^2}{16}F+\frac{\alpha^2}{4}\sigma ZF(1+\sigma Z) \\ & +\frac{\alpha^2}{16}+\frac{\alpha^2}{4}\sigma Z(1+\sigma Z)-\frac{\alpha^2}{8}Z'-\frac{\alpha^2}{4}\sigma ZZ'+\frac{\alpha^3}{32N_\parallel^2}Z'^2+\frac{\alpha 3}{16N_\parallel^2}\sigma ZZ'^2 \end{aligned}$$

But, since $\frac{\alpha}{2}\sigma ZF = -\frac{\alpha^2}{16N_\parallel^2}Z'^2 - \frac{\alpha^2}{8N_\parallel^2}Z'$

$$\left(\begin{array}{l} -\alpha\sigma ZF\left(1-N_\parallel^2-\frac{\alpha}{4}\right) - \frac{\alpha^2}{4}\sigma^2 Z^2 F + \frac{\alpha^2}{4}\sigma ZF(1+\sigma Z) \\ = \left(\frac{\alpha^2}{8N_\parallel^2}Z'^2 + \frac{\alpha^2}{4N_\parallel^2}Z'\right)\left(1-N_\parallel^2-\frac{\alpha}{4}\right) + \frac{\alpha^2}{4}\sigma^2 Z^2 F \\ = \left(\frac{\alpha^2}{8N_\parallel^2}Z'^2 + \frac{\alpha^2}{4N_\parallel^2}Z'\right)\left(1-N_\parallel^2-\frac{\alpha}{4}\right) + \frac{\alpha}{2}\sigma Z\left(-\frac{\alpha^2}{16N_\parallel^2}Z'^2 - \frac{\alpha^2}{8N_\parallel^2}Z'\right) \\ = \frac{\alpha^2}{8N_\parallel^2}Z'^2 + \frac{\alpha^2}{4N_\parallel^2}Z' - \frac{\alpha^2}{8}Z'2 - \frac{\alpha^2}{4}Z' - \frac{\alpha^3}{32N_\parallel^2}Z'^2 - \frac{\alpha^3}{16N_\parallel^2}Z' - \frac{\alpha^3}{32N_\parallel^2}Z'^2 - \frac{\alpha^3}{16N_\parallel^2}Z' \\ = \frac{\alpha^2}{8N_\parallel^2}Z'^2 + \frac{\alpha^2}{4N_\parallel^2}Z' - \frac{\alpha^2}{8}Z'2 - \frac{\alpha^2}{4}Z' - \frac{\alpha^3}{16N_\parallel^2}Z'^2 - \frac{\alpha^3}{8N_\parallel^2}Z' \end{array} \right.$$

$$\begin{aligned} = \; & -(1-N_\parallel^2)\left(1-N_\parallel^2-\frac{\alpha}{2}\right)F - (1-N_\parallel^2)^2 + \frac{\alpha}{2}(1-N_\parallel^2) - \alpha\sigma Z\left(1-N_\parallel^2-\frac{\alpha}{4}\right) \\ & -\left(1-\frac{\alpha}{4}-N_\parallel^2\right)(1-\alpha) - \frac{\alpha^2}{8N_\parallel^2}\left(1-\frac{\alpha}{4}-N_\parallel^2\right)Z'^2 + \frac{\alpha}{2}\left(1-N_\parallel^2-\frac{\alpha}{4}\right)Z' \\ & -\left(1-\frac{\alpha}{4}-N_\parallel^2\right)N_\parallel^2 - \frac{\alpha}{2}\sigma Z(1-\alpha) - \frac{\alpha}{2}N_\parallel^2\sigma Z + \frac{\alpha^2}{4}\sigma Z - \frac{\alpha^2}{8}Z' + \frac{\alpha^3}{32N_\parallel^2}Z'^2 + \frac{\alpha^2}{8N_\parallel^2}Z'^2 \\ & +\frac{\alpha^2}{4N_\parallel^2}Z' - \frac{\alpha^2}{8}Z'^2 - \frac{\alpha^2}{4}Z' - \frac{\alpha^3}{16N_\parallel^2}Z'^2 - \frac{\alpha^3}{8N_\parallel^2}Z' = \mathbf{B} \end{aligned}$$

<u>For the terms which do not contain $Z, Z', \text{and} F$</u>

$$\begin{aligned} & -(1-N_\parallel^2)^2 + \frac{\alpha}{2}(1-N_\parallel^2) - (1-\frac{\alpha}{4}-N_\parallel^2)(1-\alpha) - (1-\frac{\alpha}{4}-N_\parallel^2)N_\parallel^2 \\ = \; & -1+2N_\parallel^2 - N_\parallel^4 + \frac{\alpha}{2} - \frac{\alpha}{2}N_\parallel^2 - 1 + \alpha + \frac{\alpha}{4} - \frac{\alpha^2}{4} + N_\parallel^2 - \alpha N_\parallel^2 - N_\parallel^2 + \frac{\alpha}{4}N_\parallel^2 + N_\parallel^4 \\ = \; & -2 + 2N_\parallel^2 - \frac{5}{4}\alpha N_\parallel^2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} \\ = \; & -2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} + (2-\frac{5}{4}\alpha)N_\parallel^2 \end{aligned}$$

<u>For the coefficients of Z</u>

$$\begin{aligned} & -\sigma\alpha(1-N_\parallel^2-\frac{\alpha}{4}) - \frac{\alpha}{2}\sigma(1-\alpha) - \frac{\alpha}{2}N_\parallel^2\sigma + \frac{\alpha^2}{4}\sigma \\ = \; & -\sigma\alpha + \sigma\alpha N_\parallel^2 + \frac{\alpha^2}{4}\sigma - \frac{\alpha}{2}\sigma + \frac{\alpha^2}{2}\sigma - \frac{\alpha}{2}N_\parallel^2\sigma + \frac{\alpha^2}{4}\sigma \\ = \; & -\frac{3}{2}\sigma\alpha + \frac{1}{2}\sigma\alpha N_\parallel^2 + \sigma\alpha^2 \\ = \; & \sigma\frac{\alpha}{2}(N_\parallel^2 - 3 + 2\alpha) \end{aligned}$$

For the coefficients of Z'

$$\begin{aligned}
& \frac{\alpha^2}{4N_\parallel^2} - \frac{\alpha^2}{4} - \frac{\alpha^3}{8N_\parallel^2} + \frac{\alpha}{2}(1 - N_\parallel^2 - \frac{\alpha}{4}) - \frac{\alpha^2}{8} \\
= & \frac{\alpha^2}{4N_\parallel^2} - \frac{\alpha^3}{8N_\parallel^2} + \frac{\alpha}{2}(1 - N_\parallel^2) - \frac{\alpha^2}{4} - \frac{\alpha^2}{8} - \frac{\alpha^2}{8} \\
= & \frac{\alpha}{2}[1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2})]
\end{aligned}$$

Thus, the coefficient of $N_\perp^2$, B is

$$\begin{aligned}
B = & \sigma\frac{\alpha}{2}(N_\parallel^2 - 3 + 2\alpha)Z - 2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} + (2 - \frac{5}{4}\alpha)N_\parallel^2 \\
& + \frac{\alpha}{2}[1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2})]Z' - (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})F
\end{aligned}$$

(3) Constant term

$$\begin{aligned}
& (1 - \frac{\alpha}{4} + \frac{\alpha}{2}\sigma Z - N_\parallel^2)^2(1 - \alpha) - \frac{\alpha}{16}(1 + 4\sigma Z + 4\sigma^2 Z^2)(1 - \alpha) \\
= & (1 - \frac{\alpha}{4} - N_\parallel^2)^2(1 - \alpha) + \alpha\sigma(1 - \frac{\alpha}{4} - N_\parallel^2)(1 - \alpha)Z + \frac{\alpha^2}{4}\sigma^2 Z^2(1 - \alpha) \\
& -(\frac{\alpha^2}{16} + \frac{\alpha^2}{4}\sigma Z + \frac{\alpha^2}{4}\sigma^2 Z^2)(1 - \alpha) \\
= & (1 - \frac{\alpha}{4} - N_\parallel^2)^2(1 - \alpha) - \frac{\alpha^2}{16}(1 - \alpha) + \alpha\sigma Z(1 - \alpha)(1 - \frac{\alpha}{4} - N_\parallel^2 - \frac{\alpha}{4}) \\
= & \sigma\frac{\alpha}{2}(1 - \alpha)(2 - \alpha - 2N_\parallel^2)Z + (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})(1 - \alpha) \\
= & C
\end{aligned}$$

Therefore, $M = AN_\perp^4 + BN_\perp^2 + C = 0$
Where

$$\begin{aligned}
A = & \sigma\frac{\alpha}{2}Z + 1 - \frac{\alpha}{4} - \frac{\alpha}{2}(1 + \frac{\alpha}{4N_\parallel^2})Z' + (1 - \frac{\alpha}{4} - N_\parallel^2)F \\
B = & \sigma\frac{\alpha}{2}(N_\parallel^2 - 3 + 2\alpha)Z - 2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} + (2 - \frac{5}{4}\alpha)N_\parallel^2 \\
& + \frac{\alpha}{2}[1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2})]Z' - (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})F \\
C = & \sigma\frac{\alpha}{2}(1 - \alpha)(2 - \alpha - 2N_\parallel^2)Z + (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})(1 - \alpha)
\end{aligned}$$

Since $F \sim \frac{1}{\sigma}$ and $\sigma \gg 1$
We may neglect the terms explicitly including F in A, B, and C.

<A. I. Akhiezer>

$M = AN_\perp^4 + BN_\perp^2 + C \Rightarrow$ Regroup about Z, Z' and constants

$$
\begin{aligned}
M &= \sigma\frac{\alpha}{2}ZN_\perp^4 + (1-\frac{\alpha}{4})N_\perp^4 - \frac{\alpha}{2}(1+\frac{\alpha}{4N_\parallel^2})Z'N_\perp^4 \\
&\quad +\sigma\frac{\alpha}{2}(N_\parallel^2 - 3 + 2\alpha)ZN_\perp^2 + [-2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} + (2 - \frac{5}{4}\alpha)N_\parallel^2]N_\perp^2 \\
&\quad +\frac{\alpha}{2}[1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2})]Z'N_\perp^2 \\
&\quad +\sigma\frac{\alpha}{2}(1-\alpha)(2 - \alpha - 2N_\parallel^2)Z + (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})(1-\alpha) \\
&= [\sigma\frac{\alpha}{2}N_\perp^4 + \sigma\frac{\alpha}{2}(N_\parallel^2 - 3 + 2\alpha)N_\perp^2 + \sigma\frac{\alpha}{2}(1-\alpha)(2 - \alpha - 2N_\parallel^2)]Z \\
&\quad +(1 - \frac{\alpha}{4})N_\perp^4 + [-2 + \frac{7}{4}\alpha - \frac{\alpha^2}{4} + (2 - \frac{5}{4}\alpha)N_\parallel^2]N_\perp^2 + (1 - N_\parallel^2)(1 - N_\parallel^2 - \frac{\alpha}{2})(1-\alpha) \\
&\quad +[-\frac{\alpha}{2}(1 + \frac{\alpha}{4N_\parallel^2})N_\perp^4 + \frac{\alpha}{2}(1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2}))N_\perp^2]Z' \\
&= \sigma\frac{\alpha}{2}[N_\perp^4 - (3 - N_\parallel^2 - 2\alpha)N_\perp^2 + (1-\alpha)(2 - \alpha - 2N_\parallel^2)]Z \\
&\quad +(1 - \frac{\alpha}{4})N_\perp^4 - [2 - \frac{7}{4}\alpha + \frac{\alpha^2}{4} - (2 - \frac{5}{4}\alpha)N_\parallel^2]N_\perp^2 + (1 - N_\parallel^2)(1-\alpha)(1 - N_\parallel^2 - \frac{\alpha}{2}) \\
&\quad +[-\frac{\alpha}{2}(1 + \frac{\alpha}{4N_\parallel^2})N_\perp^4 + \frac{\alpha}{2}(1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2}))N_\perp^2]Z' \\
&= \sigma M_0 + M_1 + Z'M_2 = 0
\end{aligned}
$$

where,

$$
\begin{aligned}
M_0 &= \frac{\alpha}{2}[N_\perp^4 - (3 - N_\parallel^2 - 2\alpha)N_\perp^2 + (1-\alpha)(2 - \alpha - 2N_\parallel^2)]Z \\
M_1 &= (1 - \frac{\alpha}{4})N_\perp^4 - [2 - \frac{7}{4}\alpha + \frac{\alpha^2}{4} - (2 - \frac{5}{4}\alpha)N_\parallel^2]N_\perp^2 + (1 - N_\parallel^2)(1-\alpha)(1 - N_\parallel^2 - \frac{\alpha}{2}) \\
M_2 &= -\frac{\alpha}{2}(1 + \frac{\alpha}{4N_\parallel^2})N_\perp^4 + \frac{\alpha}{2}(1 - \alpha - N_\parallel^2 + \frac{\alpha}{2N_\parallel^2}(1 - \frac{\alpha}{2}))N_\perp^2
\end{aligned}
$$

From the equation including σ,

we take Taylor expansion at $N_{\perp 0}$ of $(\sigma M_0)_{N_{\perp 0}} = 0$

$$
\sigma M_0 = 0 \rightarrow M_0 = 0 \left(\begin{array}{l} \because M = \sigma M_0 + M_1 + M_2 Z' = 0 \\ \sigma >> 1, \ \rightarrow \underbrace{\sigma M_0 >> M_1 + M_2 Z'}_{\text{weak-damping}} \rightarrow M \approx \sigma M_0 = 0) \end{array} \right.
$$

Thus,

$$N_{\perp 0}^2 = \frac{1}{2}[(3-N_\parallel^2-2\alpha) \pm \sqrt{(3-N_\parallel^2-2\alpha)^2 - 4(1-\alpha)(2-\alpha-2N_\parallel^2)}]$$

but,
$$\begin{aligned}
&(3-N_\parallel^2-2\alpha)^2 - 4(1-\alpha)(2-\alpha-2N_\parallel^2)\\
&= 9 + N_\parallel^4 + 4\alpha^2 - 6N_\parallel^2 - 12\alpha + 4\alpha N_\parallel^2 - 8 + 4\alpha + 8N_\parallel^2 + 8\alpha - 4\alpha^2 - 8\alpha N_\parallel^2\\
&= 1 + N_\parallel^4 + 2N_\parallel^2 - 4\alpha N_\parallel^2\\
&= (1+N_\parallel^2)^2 - 4\alpha N_\parallel^2
\end{aligned}$$

$$\therefore N_{\perp 0}^2 = \frac{1}{2}\left[(3-N_\parallel^2-2\alpha) \pm \sqrt{(1+N_\parallel^2)^2 - 4\alpha N_\parallel^2}\right]$$

(+) sign : X-mode like
(−) sign : O-mode like

Next order solution

$$\sigma M_0 = -(M_1 + Z' M_2)$$

$$(\sigma M_0)_{N_{\perp 0}} + \left(\frac{\partial \sigma M_0}{\partial N_\perp}\right)_{N_\perp = N_{\perp 0}} \delta N_\perp = -(M_1 + Z' M_2)_{N_\perp = N_{\perp 0}}$$

$$\because \delta N_\perp = -\frac{M_1 + Z' M_2}{\left(\frac{\partial \sigma M_0}{\partial N_\perp}\right)}|_{N_\perp = N_{\perp 0}}$$

⇒ "the change of refractive index of wave near the electron cyclotron resonance zone when the wave propagate"

Using the identity of the Fried-Conte function

$$Z'(x) = -2[1 + xZ(x)], \text{ and}$$

$$\begin{aligned}
\frac{\partial \sigma M_0}{\partial N_\perp}|_{N_{\perp 0}} &= \sigma\frac{\alpha}{2}[4N_{\perp 0}^3 - 2(3-N_\parallel^2-2\alpha)N_{\perp 0}]Z\\
&= \sigma\frac{\alpha}{2}N_{\perp 0}[4N_{\perp 0}^2 - 6 + 2N_\parallel^2 + 4\alpha]Z\\
&= \sigma\alpha N_{\perp 0}[2N_{\perp 0}^2 - 3 + 2N_\parallel^2 + 2\alpha]Z
\end{aligned}$$

$$[M_1 + Z' M_2] = M_1 - 2[1+\zeta Z]M_2 = M_1 - 2M_2 - 2\zeta M_2 Z.$$

$$\therefore \delta N_\perp = -\frac{M_1 - 2M_2 - 2\zeta M_2 Z}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel + 2\alpha)}$$

Since $Z(\zeta)$ is complex,

$$\begin{aligned}
\delta N_\perp &= -\frac{(M_1 - 2M_2 - 2\zeta M_2 Z)(\mathrm{Re}Z - i\mathrm{Im}Z)}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2} \\
&= -\frac{[M_1 - 2M_2 - 2\zeta M_2(\mathrm{Re}Z + i\mathrm{Im}Z)](\mathrm{Re}Z - i\mathrm{Im}Z)}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2} \\
&= -\frac{2\zeta M_2|Z|^2 - (M_1 - 2M_2)\mathrm{Re}Z + i(M_1 - 2M_2)\mathrm{Im}Z}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathrm{Re}\,\delta N_\perp &= -\frac{2\zeta M_2|Z|^2 - (M_1 - 2M_2)Re\ Z}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2} \\
\mathrm{Im}\,\delta N_\perp &= -\frac{(M_1 - 2M_2)\mathrm{Im}\ Z}{\sigma\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2}
\end{aligned}$$

$$\text{But}, \sigma = \frac{R}{\sqrt{2}\Delta}\ ,\ \zeta = \frac{x}{\sqrt{2}\Delta}$$

$$Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t-\zeta}dt$$

When ζ is real,

$$\begin{aligned}
Z(\zeta) &= i\pi^{\frac{1}{2}}e^{-\zeta^2} - 2\zeta Y(\zeta) \qquad \text{where, } Y(\zeta) = \frac{1}{\zeta}e^{-\zeta^2}\int_0^\zeta e^{t^2}dt \\
\text{Then, Im } Z(\zeta) &= \sqrt{\pi}e^{-\zeta^2} = \sqrt{\pi}e^{-\frac{x^2}{2\Delta^2}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Re}\,\delta N_\perp &= -\frac{2\frac{x}{\sqrt{2}\Delta}M_2|Z|^2 - \Lambda_1(Re\ Z)}{\frac{R}{\sqrt{2}\Delta}\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2} \\
&= \frac{\sqrt{2}}{R}\Delta - \frac{(\sqrt{2}\frac{x}{\Delta})M_2|Z|^2 - \Lambda_1(Re\ Z)}{\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\,\delta N_\perp &= -\frac{\Lambda_1\sqrt{\pi}e^{\frac{-x^2}{2\Delta^2}}}{\frac{R}{\sqrt{2}\Delta}\alpha N_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2} \\
&= \frac{\sqrt{2\pi}}{\alpha}\frac{\Delta\Lambda_1 e^{\frac{-x^2}{2\Delta^2}}}{RN_{\perp 0}(2N_{\perp 0}^2 - 3 + N_\parallel^2 + 2\alpha)|Z|^2}, \quad \Delta = RN_\parallel\sqrt{\frac{T_e}{m_e c^2}}
\end{aligned}$$

:“Damping Rate of Waves near the ECR zone”

Where,

$$\begin{aligned}
\Lambda_1 &= (M_1 - 2M_2)_{N_{\perp 0}} \\
&= (1-\frac{\alpha}{4})N_{\perp 0}^4 - [2-\frac{7}{4}\alpha+\frac{\alpha^2}{4}-(2-\frac{5}{4}\alpha)N_\parallel^2]\ N_{\perp 0}^2 + (1-N_\parallel^2)(1-\alpha)(1-N_\parallel^2-\frac{\alpha}{2}) \\
&\quad +\alpha(1+\frac{\alpha}{4N_\parallel^2})N_{\perp 0}^4 - \alpha[1-\alpha-N_\parallel^2+\frac{\alpha}{2N_\parallel^2}(1-\frac{\alpha}{2})]N_{\perp 0}^2 \\
&= (1+\frac{3\alpha}{4}+\frac{\alpha^2}{4N_\parallel^2}))N_{\perp 0}^4 - [2-\frac{3\alpha}{4}-\frac{3\alpha^2}{4}+\frac{\alpha}{4}N_\parallel^2-2N_\parallel^2+\frac{\alpha^2}{2N_\parallel^2}(1-\frac{\alpha}{2})]\ N_{\perp 0}^2 \\
&\quad +(1-N_\parallel^2)(1-\alpha)(1-N_\parallel^2-\frac{\alpha}{2})
\end{aligned}$$

6.2 Damping Rates Using Quasi-linear Theory

(ref. Owen C. Eldridge and Won Namkung, ORNL/TM-6052)

The Fokker-Plank form from Quasi-linear theory (see Appendix 3.)

$$\begin{aligned}\frac{\partial f}{\partial t} &= \pi(\frac{e}{2m\omega})^2 \sum_{modes}\sum_{n=-\infty}^{\infty}\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}[n\Omega|A|^2\delta(\omega-n\Omega-k_zv_z)\\ &\times (\frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}+k_z\frac{\partial f}{\partial v_z})]+\frac{\partial}{\partial v_z}[k_z|A|^2\delta(\omega-n\Omega-k_zv_z)\\ &\times (\frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}+k_z\frac{\partial f}{\partial v_z})]\}\end{aligned}$$

with

$$\begin{aligned}A &= v_\perp E^- e^{i\theta}J_{n+1}+v_\perp E^+e^{-i\theta}J_{n-1}+2v_zE_zJ_n.\\ J_n &= J_n(\frac{k_\perp v_\perp}{\Omega})\\ E^+ &= E_x+iE_y,\ E^-=E_x-iE_y\\ B_z &= B_TR_0/(R_0+x)=B_T(R-x)/R\end{aligned}$$

where B_T is the toroidal magnetic field at cyclotron resonance and

$$n\Omega(x)=\omega(1-\frac{x}{R})$$

* $\sum_{\text{modes}}$ is the summation over all possible perturbed modes,

and $\sum_n$ is the summation over all harmonics.

For Maxwellian Distribution,

$$\begin{aligned}f &= n_e(x)[\frac{m}{2\pi T(x)}]^{3/2}\exp(-\frac{mv^2}{2T})\\ &= n_e(x)[\frac{m}{2\pi T(x)}]^{3/2}\exp(-\frac{mv_\perp^2}{2T}-\frac{mv_z^2}{2T})\\ \Rightarrow \frac{\partial f}{\partial v_\perp} &= [\frac{m}{2\pi T}]^{3/2}n_e(-\frac{mv_\perp}{T})e^{-\frac{mv^2}{2T}}\\ \frac{\partial f}{\partial v_z} &= [\frac{m}{2\pi T}]^{3/2}n_e(-\frac{mv_z}{T})e^{-\frac{mv^2}{2T}}\end{aligned}$$

Where $n_e(x)$ is the plasma density.

$$\Rightarrow \frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}+k_z\frac{\partial f}{\partial v_z}=[\frac{m}{2\pi T}]^{3/2}n_e(-\frac{m}{T})[n\Omega+k_zv_z]e^{-\frac{mv^2}{2T}}$$

When, this term is integrated over v_z, $n\Omega+k_zv_z=\omega$.
Then, above term becomes

$$\frac{n\Omega}{v_\perp}\frac{\partial f}{\partial v_\perp}+k_z\frac{\partial f}{\partial v_z}=-[\frac{m}{2\pi T}]^{3/2}\frac{n_em\omega}{T}e^{-\frac{mv^2}{2T}}$$

And, the argument of delta function is

$$\begin{aligned}\delta(\omega - n\Omega - k_z v_z) &= \delta(\omega - \omega(1-\frac{x}{R}) - k_z v_z)\\ &= (\frac{\omega x}{R} - k_z v_z)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\pi\frac{e^2 n_e(x)}{4m^2\omega^2}\frac{m\omega}{T(x)}[\frac{m}{2\pi T(x)}]^{3/2}\sum_{\text{modes}}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\right.\\ &\times\left[n\Omega(x)|A|^2\delta(\frac{\omega x}{R} - k_z v_z)\exp(-\frac{mv^2}{2T})\right]\\ &\left.+\frac{\partial}{\partial v_z}\left[k_z|A|^2\delta(\frac{\omega x}{R} - k_z v_z)\exp(-\frac{mv^2}{2T})\right]\right\}\end{aligned}$$

Since, $\omega_p^2 = \frac{e^2 n_e(x)}{m\epsilon_0}$

$$\frac{\pi e^2 n_e(x)}{4m\omega T(x)} = \pi\frac{\epsilon_0\omega_p^2}{4\omega}\frac{1}{T}$$

$$\begin{aligned}\therefore\frac{\partial f}{\partial t} &= -\pi\frac{\epsilon_0\omega_p^2}{4\omega}\frac{1}{T}[\frac{m}{2\pi T}]^{3/2}\sum_{modes}\sum_{n=-\infty}^{\infty}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\right.\\ &\times\left[n\Omega(x)|A|^2\delta(\frac{\omega x}{R} - k_z v_z)\exp(-\frac{mv^2}{2T})\right]\\ &\left.+\frac{\partial}{\partial v_z}\left[k_z|A|^2\delta(\frac{\omega x}{R} - k_z v_z)\exp(-\frac{mv^2}{2T})\right]\right\}\end{aligned}$$

Where,

$$A = v_\perp E^- e^{i\alpha}J_{n+1} + v_\perp E^+ e^{-i\alpha}J_{n-1} + 2v_z E_z J_n$$

For the small argument, the Bessel functions are expanded, but $n+1$ order Bessel function is smaller than the other two and is neglected.

$$J_n(x) \sim (\frac{1}{2}x)^n/\Gamma(n+1) = \frac{x^n}{2^n n!}(n \neq -1, -2, -3, \cdots)$$

For $n = -1, -2, -3, \cdots$ (electrons)

$$\text{Using } J_n(x) = J_{-M}(x) = (-1)^M J_M(x)$$

$$\begin{aligned}J_{n+1} &= J_{-M+1}(x)\\ &= J_{-(M-1)}(x)\\ &= (-1)^{M-1}J_{M-1}(x)\end{aligned}$$

And if $\vec{k} = (k_x, 0, k_z),\ \theta = 0$.
Then,

$$\begin{aligned}A &= v_\perp E^-\frac{(\frac{k_\perp v_\perp}{|\Omega|})^{n-1}}{2^{n-1}(n-1)!}(-1)^{n-1} + 2v_z E_z\frac{(\frac{k_\perp v_\perp}{|\Omega|})^n}{2^n n!}(-1)^n\ (n = 1, 2, 3, \cdots)\\ &= E^-\frac{k_\perp^{n-1}v_\perp^n}{(2|\Omega|)^{n-1}(n-1)!}(-1)^{n-1} + E_z\frac{2k_\perp^n v_\perp^n v_z}{(2|\Omega|)^n n!}(-1)^n\\ &= \frac{k_x^{n-1}v_\perp^n}{(2|\Omega|)^{n-1}(n-1)!}\left(E^- + \frac{k_x v_z E_z}{n|\Omega|}\right)\end{aligned}$$

Thus,

$$|A|^2 = \frac{(k_x^2)^{n-1} v_\perp^{2n}}{(4\Omega^2)^{n-1}[(n-1)!]^2} \left|E^- + \frac{k_x v_z E_z}{n|\Omega|}\right|^2$$

The perpendicular and parallel heating rates per unit volume are found by integrating over the velocities,

$$\frac{d^2 W_\perp}{dtdV} = \int_{-\infty}^{\infty} dv_z \int_0^\infty \underbrace{\frac{1}{2} m v_\perp^2}_{\text{non-relativistic energy}} \frac{\partial f}{\partial t} 2\pi v_\perp dv_\perp = m\pi \int\int v_\perp^3 \frac{\partial f}{\partial t} dv_\perp dv_z$$

$$\frac{d^2 W_\parallel}{dtdV} = \int_{-\infty}^{\infty} dv_z \int_0^\infty \underline{\frac{1}{2} m v_\parallel^2} \frac{\partial f}{\partial t} 2\pi v_\perp dv_\perp = m\pi \int\int v_\parallel^2 v_\perp \frac{\partial f}{\partial t} dv_\perp dv_z$$

For an energy flux $\vec{S}$ in a plane plasma, one has

$$\vec{\nabla} \cdot \vec{S} = \frac{\partial S_x}{\partial x} = -\frac{d^2 W}{dtdV} = I_m(-2k_x) S_x$$

$$\frac{1}{S_x} \frac{\partial S_x}{\partial x} = -\frac{1}{S_x} \frac{d^2 W}{dtdV} = I_m(-2k_x)$$

The total energy absorbed in the resonant surface.

$$\begin{aligned} W_{abs} &= W_0 (1 - e^{\int_{-\infty}^{\infty} dx I_m(-2k_x)}) \\ &= W_0 (1 - e^{-\eta}), \qquad \text{where } \eta = \int_{-\infty}^{\infty} dx I_m(2k_x) \end{aligned}$$

1) Integration of $\frac{d^2 W_\perp}{dtdv}$,

$$\frac{d^2 W_\perp}{dtdV} = -m\pi \frac{\omega_p^2}{16\omega} \frac{1}{T(x)} \left[\frac{m}{2\pi T(x)}\right]^{\frac{3}{2}} \int_{-\infty}^{\infty} dv_z \int_0^\infty v_\perp^3 \Big\{ \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \times \underbrace{[n|\Omega||A|^2 \delta(\frac{\omega x}{R} - k_z v_z) e^{-\frac{mv^2}{2T}}]}_{1} + \frac{\partial}{\partial v_z} \underbrace{[k_z |A|^2 \delta(\frac{\omega x}{R} - k_z v_z) e^{-\frac{mv^2}{2T}}]}_{2} \Big\} dv_\perp$$

$$\begin{aligned} 1) \int_{-\infty}^{\infty} dv_z \int_0^\infty v_\perp^3 \frac{\partial}{\partial v_z} [2]\, dv_\perp &= \int_0^\infty v_\perp^3 \int_{-\infty = v_z}^{\infty = v_z} d[2]\, dv_\perp \\ &= \int_0^\infty v_\perp^3 \left(k_z |A|^2 \delta(\frac{\omega x}{R} - k_z v_z) e^{-\frac{mv^2}{2T}} \right) dv_\perp = 0 \end{aligned}$$

$$\begin{aligned} 2) \frac{\partial}{\partial v_\perp} \left[n\Omega^2 |A|^2 \delta(\frac{\omega x}{R} - k_z v_z) e^{-\frac{mv^2}{2T}} \right] &= n|\Omega| \frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}[(n-1)!]^2} \delta(\frac{\omega x}{R} - k_z v_z) \\ &\times \left[2n(v_\perp)^{2n-1} - \frac{m(v_\perp)^{2n+1}}{T} \right] e^{-\frac{mv^2}{2T}} \left|E^- + \frac{k_x v_z E_z}{n|\Omega|}\right|^2 \end{aligned}$$

$$= n|\Omega| \frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}[(n-1)!]^2} \; \delta(\frac{\omega x}{R} - k_z v_z) 2n (v_\perp)^{2n-1} \left(1 - \frac{m}{2nT} v_\perp^2\right) e^{-\frac{mv^2}{2T}}$$
$$\times \; |E^- + \frac{k_x v_z E_z}{n|\Omega|}|^2$$

Then, the integration gives

$$\int_0^\infty \left[\int_{-\infty}^\infty dv_z \delta(\frac{\omega x}{R} - k_z v_z) \; \left|E^- + \frac{k_x v_z E_z}{n|\Omega|}\right|^2 \; e^{-\frac{mv_z^2}{2T}} \right]$$
$$\times \; n|\Omega| \frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}[(n-1)!]^2} 2n (v_\perp^{2n+1} - \frac{m}{2nT} v_\perp^{2n+3}) e^{-\frac{mv_\perp^2}{2T}} dv_\perp$$

$$= \frac{2n^2|\Omega|}{k_z} \frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}[(n-1)!]^2} \left| E^- + \frac{k_x \omega x E_z}{k_z R n|\Omega|} \right|^2 \; \exp(-\frac{m}{2T} \frac{\omega^2 x^2}{k_z^2 R^2})$$
$$\times \; \underbrace{\int_0^\infty (\underbrace{v_\perp^{2n+1}}_a - \underbrace{\frac{m}{2nT} v_\perp^{2n+3}}_b) \; e^{-\frac{mv_\perp^2}{2T}} dv_\perp}_3$$

3) Calculation of 3

$$a) \quad \int_0^\infty v_\perp^{2n+1} e^{-\frac{mv_\perp^2}{2T}} dv_\perp \qquad \left(\int_0^\infty e^{-t} t^z dt = Z! \right)$$

$$Let, \quad \frac{mv_\perp^2}{2T} = t \; , \quad v_\perp = (\frac{2T}{m} t)^{1/2}, \quad dv_\perp = \frac{1}{2} (\frac{2T}{m} t)^{-1/2} (\frac{2T}{m}) dt$$

$$\Rightarrow \int_0^\infty (\frac{2T}{m} t)^{n+\frac{1}{2}} \; \frac{1}{2} (\frac{2T}{m} t)^{-1/2} (\frac{2T}{m}) e^{-t} dt$$

$$= \frac{1}{2} \left(\frac{2T}{m}\right)^{n+1} \int_0^\infty t^n e^{-t} dt = \frac{1}{2} \left(\frac{2T}{m}\right)^{n+1} n!$$

$$b) \quad \frac{m}{2nT} \int_0^\infty v_\perp^{2n+3} e^{-\frac{mv_\perp^2}{2T}} dv_\perp \quad = \quad \frac{m}{2nT} \frac{1}{2} \left(\frac{2T}{m}\right)^{n+2} \int_0^\infty t^{n+1} e^{-t} dt$$
$$= \quad \frac{m}{4nT} \left(\frac{2T}{m}\right)^{n+2} (n+1)!$$

(a)-(b)

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n! \; - \; \frac{m}{4nT}\left(\frac{2T}{m}\right)^{n+2}(n+1)! \\
&= \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n! \; - \; \frac{1}{2n}\frac{m}{2T}\left(\frac{2T}{m}\right)^{n+2}(n+1)! \\
&= \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n! \; - \; \frac{1}{2n}\left(\frac{2T}{m}\right)^{n+1}(n+1)! \\
&= \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n!\left(1-\frac{n+1}{n}\right) \; = \; \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1}(n)!\left(\frac{-1}{n}\right) \\
&= -\frac{1}{2}\left(\frac{2T}{m}\right)^{n+1}(n-1)!
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{d^2W_\perp}{dtdV} = &-\frac{m\pi}{k_z}\frac{\epsilon_0\omega_p^2}{4\omega}\frac{1}{T(x)}\left[\frac{m}{2\pi T(x)}\right]^{\frac{3}{2}} 2n^2|\Omega|\frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}[(n-1)!]^2} \\
&\times\left(-\frac{1}{2}\right)\left(\frac{2T}{m}\right)^{n+1}(n-1)!\left|E^- + \frac{k_x v_z E_z}{n|\Omega|}\right|^2 \exp\left(-\frac{m}{2T}\frac{\omega^2x^2}{k_z^2R^2}\right)
\end{aligned}
$$

since, $k_x = \frac{\omega}{c}N_x, \quad k_z = \frac{\omega}{c}N_z$

- $$\frac{(k_x^2)^{n-1}}{(4\Omega^2)^{n-1}} = \left(\frac{1}{4\Omega^2}\frac{\omega^2}{c^2}N_x^2\right)^{n-1} \simeq \left(\frac{N_x^2}{4\Omega^2}\frac{n^2\omega^2}{c^2}\right)^{n-1} = \left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}$$

- $$-\frac{m}{2T}\frac{\omega^2x^2}{k_z^2R^2} = -\frac{m}{2T}\frac{\omega^2x^2}{R^2}\frac{c^2}{\omega^2N_z^2} = -\frac{1}{2}\frac{mc^2}{T}\frac{1}{N_z^2R^2}x^2 = -\frac{x^2}{2\Delta^2}$$

$$\text{where,} \quad \Delta^2 = N_z^2R^2\left(\frac{T}{mc^2}\right)$$

- $$-\frac{k_x\omega x E_z}{k_z R n|\Omega|} = \frac{N_x\omega x E_z}{N_z R n|\Omega|} \simeq \frac{N_x}{N_z}\frac{x}{R}E_Z$$

$$
\begin{aligned}
\therefore \qquad \frac{d^2W_\perp}{dtdv} = &\frac{m\pi}{k_zT}\frac{\epsilon_0\omega_p^2}{4\omega}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} n^2|\Omega|\left(\frac{n^2n_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^{n+1} \\
&\times\left|E^- + \frac{n_x}{n_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}
\end{aligned}
$$

2) Integration of $\frac{d^2V_\parallel}{dtdV}$

$$\frac{d^2V_\parallel}{dtdV} = m\pi\int_{-\infty}^{\infty} dv_z \int_{\infty}^{\infty} dv_\perp v_z^2 v_\perp \frac{\partial f}{\partial t} = -m\pi \frac{\epsilon_0\omega_p^2}{4\omega}\frac{1}{T}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}}$$

$$\times \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_\perp v_z^2 v_\perp \left[\frac{1}{v_\perp}\left\{\frac{\partial}{\partial v_\perp} n|\Omega||A|^2 \delta\left(\frac{\omega_x}{R} - R_z v_z\right) e^{-\frac{mv^2}{2T}}\right\}\right.$$

$$\left. + \frac{\partial}{\partial v_z}\left\{k_z |A|^2 \delta\left(\frac{\omega_x}{R} - k_z v_z\right) e^{-\frac{mv^2}{2T}}\right\}\right]$$

(a) The first term of the integrand

$$\int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp v_z^2 \left[n|\Omega|\left(\frac{n^2 n_x^2}{4c^2}\right)^{n-1} \frac{1}{[(n-1)!]^2}\delta\left(\frac{\omega x}{R} - R_z v_z\right)\right.$$

$$\left. \times 2n(v_\perp)^{2n-1}\left(1 - \frac{m}{2nT}v_\perp^2\right) \times e^{-\frac{mv^2}{2T}}\right] \times \left|E^- + \frac{n_x}{n_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2$$

$$= \frac{1}{k_z}\left(\frac{\omega x}{k_z R}\right)\left|E^- + \frac{n_x}{n_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}$$

$$\times \int_0^{\infty} n|\Omega|\left(\frac{n^2 n_x^2}{4c^2}\right)^{n-1}\frac{1}{[(n-1)!]^2} \times 2n(v_\perp)^{2n-1}\left(1 - \frac{m}{2nT}v_\perp^2\right) e^{-\frac{mv^2}{2T}} \mathrm{d}v_\perp$$

But,

$$\int_0^{\infty} v_\perp^{2n-1} e^{-\frac{mv_\perp^2}{2T}} dv_\perp = \frac{1}{2}\left(\frac{2T}{m}\right)^n (n-1)!$$
$$\int_0^{\infty} v_\perp^{2n+1} e^{-\frac{mv_\perp^2}{2T}} dv_\perp = \frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n!$$

$$= \frac{1}{k_z}\left(\frac{\omega_x}{k_z R}\right)\left|E^- + \frac{n_x}{n_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}} 2n^2|\Omega|\left(\frac{n^2 n_x^2}{4c^2}\right)^{n-1}\frac{1}{\{(n-1)!\}^2}$$

$$\times \left(\frac{1}{2}\left(\frac{2T}{m}\right)^n (n-1)! - \frac{m}{2nT}\frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n!\right) = 0$$

(b) The second term of the integrand

$$\int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_\perp v_z^2 v_\perp \frac{\partial}{\partial v_z}\{k_z|A|^2\delta\left(\frac{\omega_x}{R} - k_z v_z\right) e^{-\frac{mv^2}{2T}}\}$$
$$= \int_{-\infty}^{\infty} dv_\perp v_\perp \left| v_z^2 k_z |A|^2 \delta\left(\frac{\omega_x}{R} - k_z v_z\right) e^{-\frac{mv^2}{2T}}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dv_z 2 v_z k_z |A|^2 \delta\left(\frac{\omega_x}{R} - k_z v_z\right) e^{-\frac{mv^2}{2T}}$$
$$= \int_{-\infty}^{\infty} dv_\perp v_\perp e^{-\frac{mv^2}{2T}} \times \left[(-2)\frac{\omega x}{k_z R}|A|^2 e^{-\frac{x^2}{2\Delta^2}}\right]$$
$$= -2\left(\frac{\omega_x}{k_z R}\right) e^{-\frac{x^2}{2\Delta^2}} \left(\frac{n^2 n_x^2}{4c^2}\right)^{n-1} \frac{1}{[(n-1)!]^2} \left|E^- + \frac{n_x}{n_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 \times \underbrace{\int_0^{\infty} v_\perp^{2n+1} e^{-\frac{mv_\perp^2}{2T}} dv_\perp}_{\frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n!}$$
$$= -2\frac{\omega x}{k_z R}\left(\frac{n^2 N_x^2}{4c^2}\right)^{n-1} \frac{1}{[(n-1)!]^2}\frac{1}{2}\left(\frac{2T}{m}\right)^{n+1} n(n-1)! \times \left|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}$$
$$= -\frac{\omega x}{k_z R}\left(\frac{n^2 N_x^2}{4c^2}\right)^{n-1} \frac{n}{(n-1)!}\left(\frac{2T}{m}\right)^{n+1} \left|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}$$

$$\therefore \frac{\mathrm{d}^2 W_\parallel}{\mathrm{d}t\mathrm{d}V}$$

$$= -m\pi\frac{\epsilon_0\omega_p^2}{4\omega}\frac{1}{T}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\left(-\frac{\omega x}{k_z R}\right)\left(\frac{n^2 N_x^2}{4c^2}\right)^{n-1}\frac{n}{(n-1)}\left(\frac{2T}{m}\right)^{n+1}\left|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}$$

$$= \frac{m\pi}{T}\frac{\epsilon_0\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\frac{\omega x}{k_z R}\frac{n}{(n-1)!}\left(\frac{n^2 N_x^2}{4c^2}\right)^{n-1}\left(\frac{2T}{m}\right)^{n+1}\left|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 e^{-\frac{x^2}{2\Delta^2}}$$

Thus,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t\mathrm{d}V} = \frac{\mathrm{d}^2 W_\perp}{\mathrm{d}t\mathrm{d}V} + \frac{\mathrm{d}^2 W_\parallel}{\mathrm{d}t\mathrm{d}V} = \frac{\mathrm{d}^2 W_\perp}{\mathrm{d}t\mathrm{d}V}\left[1 + \frac{x}{R}\right]$$

$$= \frac{m\pi}{T}\frac{\epsilon_0\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\frac{n^2|\Omega|}{(n-1)!}\left(\frac{n^2 N_x^2}{4c^2}\right)^{n-1}\left(\frac{2T}{m}\right)^{n+1}$$

$$\times \left|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z\right|^2 \left[1 + \frac{x}{R}\right] e^{-\frac{x^2}{2\Delta^2}}$$

6.2.1 Higher Harmonics ($n \geq 2$)

Let us calculate

$$\left|E^- + \tfrac{N_x}{N_z}\tfrac{x}{R}\tfrac{\omega}{n|\Omega|}E_z\right|^2$$

The nonzero components of the dielectric tensor in a cold electron plasma are

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = S$$

$$\epsilon_{xy} = -\epsilon_{yx} = i\frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}\frac{\Omega_e}{\omega} = -iD$$

and

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega} = P$$

From the dispersion relation

$$\vec{N} \times (\vec{N} \times \vec{E}) + \overleftrightarrow{\epsilon} \cdot \vec{E} = 0 \quad (\vec{N} = \frac{\vec{k}c}{\omega})$$

$$\longrightarrow \begin{pmatrix} S - N_z^2 & -iD & N_x N_z \\ iD & S - N^2 & D \\ N_x N_z & D & P - N_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(S - N_z^2)E_x - iDE_y + N_x N_z E_z = 0 \qquad (1)$$

$$iDE_x + (S - N^2)E_y = 0 \qquad (2)$$

$$N_x nz E_x + (P - N_x^2)E_z = 0 \qquad (3)$$

From equation (2)

$$i(iDE_x) + i(S - N^2)E_y = 0$$

$$-DE_x + (S - N^2)iE_y = 0$$

$$DE_x - (S - N^2)iE_y = 0 \qquad \longrightarrow E_x = \frac{S - N^2}{D} iE_y$$

From equation (3)

$$E_z = \frac{-N_x N_z E_x}{P - N_x^2} = -\frac{N_x N_z}{P - N_x^2}\frac{S - N^2}{D} iE_y$$

$$E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z = E_x - iEy + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z$$

$$= \frac{S - N^2}{D} iE_y - iE_y - \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}\frac{N_x N_z}{P - N_x^2}\frac{S - D^2}{D} iE_y$$

$$= \left(\frac{S - N^2}{D} - 1\right) iE_y - \frac{x}{R} N_x^2 \frac{\omega}{n|\Omega|(P - N_x^2)}\frac{S - N^2}{D} iE_y$$

$$= \left(\frac{S - D - N^2}{D} - \frac{x}{R} N_x^2 \frac{\omega}{n|\Omega|(P - N_x^2)}\frac{S - N^2}{D}\right) iE_y$$

$$\backsimeq \frac{S - D - N^2}{D} iE_y$$

$$\therefore \qquad \left| E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z \right|^2 = \frac{(S-D-N^2)^2}{D^2}|E_y|^2$$

But,

$$\begin{aligned} S_x &= \frac{1}{4\mu_0}\mathrm{Re}\left(\vec{E}\times\vec{B}^* + \frac{1}{2}\vec{E}^*\cdot\frac{\partial \overleftrightarrow{\epsilon}}{\partial \vec{N}}\cdot\vec{E}\right)_x \rightarrow \mathit{Ref} : \mathit{T.H.Stix},\ \mathit{page}\ 74\ (\mathit{Eqs.}(18)\ \&\ (19)) \\ &\simeq \frac{1}{4\mu_0 c}N_x|E_y|^2\frac{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}{D^2(P-N_x^2)^2} \end{aligned}$$

Detailed calculation steps are seen in Appendix 1.

* For fundamental harmonic heating, $\overleftrightarrow{\epsilon}$ is the dielectric tensor in hot plasmas. But for higher harmonic heating, $\overleftrightarrow{\epsilon}$ is the cold plasma dielectric tensor. (see Appendix 2)

$$\begin{aligned} \frac{1}{S_x}\frac{\partial S_x}{\partial x} &= -\frac{1}{S_x}\frac{\mathrm{d}^2 W}{\mathrm{d}t\mathrm{d}V} = -\frac{1}{S_x}\left(1+\frac{x}{R}\right)\frac{\mathrm{d}^2 W_\perp}{\mathrm{d}t\mathrm{d}V} \\ &= -4\mu_0 c\frac{1}{N_x}\frac{1}{|E_y|^2}\frac{D^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\frac{m\pi}{k_z T}\frac{\epsilon_0\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}} \\ &\quad \times n^2|\Omega|\left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^n|E_y|^2\frac{(S-D-N^2)^2}{D^2}e^{-\frac{x^2}{2\Delta^2}}\left(1+\frac{x}{R}\right) \\ &= -\frac{4}{c}\frac{m\pi}{k_z T}\frac{\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\frac{n^2|\Omega|}{N_x}\left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^n \\ &\quad \times \frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}e^{-\frac{x^2}{2\Delta^2}}\left(1+\frac{x}{R}\right) \\ &= I_m(-2k_x) \end{aligned}$$

Let $\eta(n) = \int_{-\infty}^{\infty} I_m(2k_x)\mathrm{d}x$: Optical depth

$$
\begin{aligned}
\therefore\ \eta(n) &= \frac{4}{c}\frac{m\pi}{k_zT}\frac{\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\frac{n^2|\Omega|}{N_x}\left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^{n+1} \\
&\times\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right]\underbrace{\int_{-\infty}^{\infty}e^{-\frac{x^2}{2\Delta^2}}\left(1+\frac{x}{R}\right)\mathrm{d}x}_{\sqrt{2\Delta^2\pi}=\Delta\sqrt{2\pi}=N_zR\sqrt{\frac{2\pi T}{mc^2}}} \\
&= \frac{4}{c}\frac{m\pi}{k_zT}\frac{\omega_p^2}{4\omega}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\frac{n^2|\Omega|}{N_x}\left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^{n+1}N_zR\sqrt{\frac{2\pi T}{mc^2}} \\
&\times\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right] \\
&= \frac{4}{c}\frac{m\pi}{k_zT_c}\frac{\omega_p^2}{4\omega^2}\left[\frac{m}{2\pi T}\right]^{\frac{3}{2}}\sqrt{\frac{2\pi T}{mc^2}}\omega\frac{n^2|\Omega|}{N_x^2}N_x\left(\frac{n^2N_x^2}{4c^2}\right)^{n-1}\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^{n}N_zR \\
&\times\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right] \\
&= \pi\frac{m}{k_zTc}\alpha\frac{m}{2\pi T}\sqrt{\frac{m}{2\pi T}}\sqrt{\frac{2\pi T}{m}}\frac{1}{c}\omega\frac{n^2|\Omega|}{N_x^2}N_x\left(\frac{n^2N_x^2}{4c^2}\right)^{n-2}\left(\frac{n^2N_x^2}{4c^2}\right) \\
&\times\frac{1}{(n-1)!}\left(\frac{2T}{m}\right)^{n-2}\left(\frac{2T}{m}\right)^{2}\left(\frac{2T}{m}\right)\frac{k_zc}{\omega}R\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right] \\
&= \pi\alpha\frac{1}{k_zc^2}\frac{1}{2\pi}n^3\omega\frac{n|\Omega|}{c^2}N_x\frac{1}{(n-1)!}\left(\frac{n^2N_x^2T}{2mc^2}\right)^{n-2}\left(\frac{2T}{m}\right)\frac{k_zc}{\omega}R \\
&\times\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right] \\
&= 2\pi\alpha\frac{T}{mc^2}\frac{n^3}{(n-1)!}\left(\frac{n^2N_x^2T}{2mc^2}\right)^{n-2}\frac{R}{\lambda}N_x\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right]
\end{aligned}
$$

Thus the fractional absorbed energy by particles by only ray passing through the resonance region is given by

$$W_{abs} = fW_0 \ (W_0 \text{ is initial energy}) = W_0 - W_0e^{-\eta}$$

$$
\begin{aligned}
&\therefore f = 1-e^{-\eta} \\
&\quad = 1-\exp\left[-2\pi\alpha\frac{R}{\lambda}\frac{T}{mc^2}\left(\frac{n^2N_x^2T}{2mc^2}\right)^{n-2}N_x\left[\frac{(S-D-N^2)^2(P-N_x^2)^2}{D^2(P-N_x^2)^2+(S-N^2)^2PN_z^2}\right]\right]
\end{aligned}
$$

Where, $\alpha = \frac{\omega_p^2}{\omega^2}$, n is the harmonic number ($n \geq 2$), T is the electron temperature, R is the scale length of the tokamak, and λ is the wavelength in free space. S, P, D is evaluated at $\omega = n|\Omega|$. The perpendicular index of refraction is

$$N_x^2 = \frac{-B \pm (B^2 - 4AC)^{\frac{1}{2}}}{2A} \quad \begin{pmatrix} + & : & \text{O-mode} \\ - & : & \text{X-mode} \end{pmatrix}$$

with

$$\begin{aligned} A &= S \\ B &= -(S+P)(S-N_z^2) + D^2 \\ C &= P[(S-N_z^2)^2 - D^2] \end{aligned}$$

6.2.2 Fundamental Harmonic ($n = 1$)

- The perpendicular heating rate per unit volume

$$\frac{d^2W_\perp}{dtdV} = \frac{m\pi}{K_zT}\frac{\epsilon_o\omega_p^2}{4\omega}[\frac{m}{2\pi T}]^{\frac{3}{2}}n^2|\Omega|(\frac{n^2N_x^2}{4c^2})^{n-1}$$
$$\frac{1}{(n-1)!}(\frac{2T}{m})^{n+1}\times|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z|^2 e^{\frac{x^2}{2\Delta^2}}$$

For $n = 1$, ($\Delta = \sqrt{\frac{k_z^2R^2T}{m\omega^2}}$)

$$\begin{aligned}
\frac{d^2W_\perp}{dtdV} &= \frac{m\pi}{k_zT}\frac{\epsilon_o\omega_p^2}{4\omega}[\frac{m}{2\pi T}]^{\frac{3}{2}}|\Omega|(\frac{2T}{m})^2\times|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2e^{-\frac{x^2}{2\Delta^2}} \\
&= \frac{m\pi}{k_zT}\frac{\epsilon_o}{4\omega}[\frac{m}{2\pi T}]^{\frac{3}{2}}(\frac{2T}{m})^2\omega_p^2|\Omega||E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2e^{-\frac{x^2}{2\Delta^2}} \\
&= \epsilon_o\sqrt{\frac{m^2\pi^2}{k_z^2T^2}\frac{1}{16\omega^2}(\frac{m}{2\pi T})^3\frac{16T^4}{m^4}}\omega_p^2|\Omega||E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2e^{-\frac{x^2}{2\Delta^2}} \\
&= \epsilon_o\sqrt{\frac{m\omega^2}{k_z^2\omega^4T}(\frac{1}{4})}\frac{1}{\sqrt{2\pi}}\omega_p^2\Omega|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2e^{-\frac{x^2}{2\Delta^2}} \\
&= \frac{\epsilon_o\omega_p^2}{2}\frac{|\Omega|}{\omega^2}\frac{R}{\Delta}\frac{e^{-\frac{x^2}{2\Delta^2}}}{\sqrt{2\pi}}|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2
\end{aligned}$$

- The parallel heating rate per unit volume

$$\frac{d^2W_\parallel}{dtdV} = \frac{m\pi}{k_zT}\frac{\epsilon_o\omega_p^2}{4\omega}[\frac{m}{2\pi T}]^{\frac{3}{2}}\frac{n^2|\Omega|}{(n-1)!}(\frac{n^2N_x^2}{4c^2})^{n-1}(\frac{2T}{m})^{n+1}|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{n|\Omega|}E_z|^2\times\frac{x}{R}e^{-\frac{x^2}{2\Delta^2}}$$

For $n = 1$, ($\Delta = \sqrt{\frac{k_z^2R^2T}{m\omega^2}}$)

similarly

$$\frac{d^2W_\parallel}{dtdV} = \frac{\epsilon_o\omega_p^2}{2}\frac{|\Omega|}{\omega^2}\frac{x}{\Delta}\frac{e^{-\frac{x^2}{2\Delta^2}}}{\sqrt{2\pi}}|E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2$$

$$\text{So, } |E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z|^2 = ?$$

From the Hot Plasma dispersion relation,

$$\begin{pmatrix} \epsilon_{xx} - N_z^2 & \epsilon_{xy} & \epsilon_{xz} + N_xN_z \\ -\epsilon_{xy} & \epsilon_{xx} - N^2 & \epsilon_{yz} \\ \epsilon_{xz} + N_xN_z & -\epsilon_{yz} & \epsilon_{zz} - N_x^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$
\begin{aligned}
(\epsilon_{xx} - N_z^2)E_x + \epsilon_{xy}E_y + (\epsilon_{xz} + N_xN_z)E_z &= 0 \\
-\epsilon_{xy}E_x + (\epsilon_{xx} - N^2)E_y + \epsilon_{yz}E_z &= 0 \\
(\epsilon_{xz} + N_xN_z)E_x - \epsilon_{yz}E_y + (\epsilon_{zz} - N_x^2)E_z &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E_z &= -\frac{(\epsilon_{xz} + N_xN_z)E^- + iN_zN_zE_y}{\epsilon_{zz} - N_x^2} \\
E^- &= E_x - iE_y \\
&= -i\frac{(\epsilon_{xx} - i\epsilon_{xy} - N_z^2)E_y - i(\epsilon_{xz} + N_xN_z)E_z}{\epsilon_{xx} - N_z^2}
\end{aligned}
$$

Note $\epsilon_{yz} = \epsilon_{zy} = i\epsilon_{xz}$ for the first-order approximation

$$
\begin{aligned}
E^- + \frac{N_x}{N_z}\frac{x}{R}\frac{\omega}{|\Omega|}E_z &= -i\frac{(\epsilon_{xx} - i\epsilon_{xy} - N_z^2)E_y - i(\epsilon_{xz} + N_xN_z)E_z}{\epsilon_{xx} - N_z^2} \\
&\quad -\frac{N_x}{N_z}\frac{x}{R}\frac{(\epsilon_{xz} + N_xN_z)E^- + iN_xN_zE_y}{\epsilon_{zz} - N_x^2} \\
&= -iE_y\left[\frac{\epsilon_{xx} - i\epsilon_{xy} - N_z^2 + N_x^2\frac{x}{R}\frac{\epsilon_{xx}-N_z^2}{\epsilon_{zz}-N_x^2}}{\epsilon_{xx} - N_z^2}\right] \\
&\quad -\frac{(\epsilon_{xz} + N_xN_z)E_z}{\epsilon_{xx} - N_z^2} - \frac{N_x}{N_z}\frac{x}{R}\frac{\epsilon_{xz} + N_xN_z}{\epsilon_{zz} - N_x^2}E^-
\end{aligned}
$$

* E^- is smaller than E_y, E_z, by a factor of $(\frac{\Delta}{R})$

$$
\begin{aligned}
\epsilon_{xx} - i\epsilon_{xy} &\simeq 1 - \frac{\omega_p^2}{2\omega^2}(\frac{\omega}{\omega + \Omega} - \frac{R}{\sqrt{2}\Delta}Z) - \frac{\omega_p^2}{2\omega^2}(\frac{\omega}{\omega + \Omega} + \frac{R}{\sqrt{2}\Delta}Z) \\
&\simeq 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega + \Omega} \sim \text{ order of 1}
\end{aligned}
$$

$$
\epsilon_{xx} \sim \text{order of } \frac{R}{\Delta}
$$

$$
\begin{aligned}
E^- + \frac{N_x}{N_z}\frac{x}{R}E_z &\simeq -iE_y\left(\frac{1}{\epsilon_{xx} - N_z^2}\right)\left(\epsilon_{xx} - i\epsilon_{xy} - N_z^2 + N_x^2\frac{x}{R}\frac{\epsilon_{xx} - N_z^2}{\epsilon_{zz} - N_x^2}\right) \\
&\quad + \frac{\epsilon_{xz} + N_xN_z}{\epsilon_{xx} - N_z^2}\frac{(\epsilon_{xz} + N_xN_z)E^- + iN_xN_zE_y}{\epsilon_{zz} - N_x^2} \\
&\simeq -iE_y\frac{1}{\epsilon_{xx} - N_z^2}\left[\epsilon_{xx} - i\epsilon_{xy} - N_z^2 - \frac{N_xN_z(\epsilon_{xz} + N_xN_z)}{\epsilon_{zz} - N_x^2} + N_x^2\frac{x}{R}\frac{\epsilon_{xx} - N_z^2}{\epsilon_{zz} - N_x^2}\right]
\end{aligned}
$$

$$\left(\begin{array}{ll}\epsilon_{xx} & = 1 - \frac{\alpha}{4} + \frac{\alpha}{2}\frac{R}{\sqrt{2}\Delta}Z \\ \epsilon_{xy} & = -\frac{i\alpha}{2}(\frac{1}{2} + \frac{R}{\sqrt{2}\Delta}Z) = -\frac{i\alpha}{4} - \frac{i\alpha R}{2\sqrt{2}\Delta}Z \\ \epsilon_{zz} & = 1 - \alpha(1 + \frac{\alpha}{4}\frac{N_x^2}{N_z^2}\frac{x}{R}Z') \\ & = 1 - \alpha[1 + \frac{\alpha}{4}\frac{N_x^2}{N_z^2}\frac{x}{R}(-2)(1 + \frac{x}{\sqrt{2}\Delta}Z)] \\ & = 1 - \alpha + \frac{\alpha^2}{4}\frac{N_x^2}{N_z^2}\frac{x}{R}(2 + \frac{2x}{2\sqrt{2}\Delta}Z) \\ \epsilon_{xz} & = \frac{-\alpha}{4}\frac{N_x}{N_z}Z' = \frac{\alpha}{4}\frac{N_x}{N_z}(2)(1 + \frac{x}{\sqrt{2}\Delta}Z)\end{array}\right)$$

Where $\alpha = \frac{\omega_p^2}{\Omega^2} \simeq \frac{\omega_p^2}{\omega^2}$

$$E^- + \frac{N_x}{N_z}\frac{x}{R}E_z \simeq -iE_y(\frac{1}{\epsilon_{xx} - N_z^2})[\epsilon_{xx} - i\epsilon_{xy} - N_z^2 + \frac{-N_xN_z\epsilon_{xz} + N_x^2\frac{x}{R}\epsilon_{xx} - N_x^2N_z^2(1 + \frac{x}{R})}{\epsilon_{zz} - N_x^2}]$$

$$\begin{aligned} -N_xN_z\epsilon_{xz} + N_x^2\frac{x}{R}\epsilon_{xx} &= N_x^2\frac{x}{R}(1 - \frac{\alpha}{4} + \frac{\alpha}{2}\frac{R}{\sqrt{2}\Delta}Z) - N_xN_z\frac{\alpha}{2}\frac{N_x}{N_z}(1 + \frac{x}{\sqrt{2}\Delta}Z) \\ &= N_x^2\frac{x}{R} - \frac{\alpha}{4}N_x^2\frac{x}{R} - N_x^2\frac{\alpha}{2} \simeq -N_x^2\frac{\alpha}{2}\end{aligned}$$

$$\begin{aligned} \epsilon_{xx} - i\epsilon_{xy} - N_z^2 &= 1 - \frac{\alpha}{4} + \frac{\alpha}{2}\frac{R}{\sqrt{2}\Delta}Z + i(\frac{i\alpha}{4} + \frac{i\alpha R}{2\sqrt{2}\Delta}Z) - N_z^2 \\ &= 1 - \frac{\alpha}{4} - \frac{\alpha}{4} - N_z^2 \\ &= 1 - \frac{\alpha}{2}N_z^2 \\ \epsilon_{zz} - N_x^2 &= 1 - \alpha + \frac{\alpha^2}{4}\frac{N_x^2}{N_z^2}\frac{x}{R}(2 + \frac{2x}{\sqrt{2}\Delta}Z) - N_x^2 \\ &\simeq 1 - \alpha - N_x^2 \\ \epsilon_{xx} - N_z^2 &= 1 - \frac{\alpha}{4} + \frac{\alpha}{2}\frac{R}{\sqrt{2}\Delta}Z - N_z^2 \\ &\simeq \frac{1}{2\sqrt{2}}\frac{\alpha R}{\Delta}Z \qquad (\because 1 - \frac{\alpha}{4} - N_z^2 \ll \frac{\alpha R}{2\sqrt{2}\Delta}Z)\end{aligned}$$

$$\begin{aligned} \therefore E^- + \frac{x}{R}\frac{N_x}{N_z}E_z &= -iE_y\frac{2\sqrt{2}\Delta}{\alpha RZ}\frac{(1 - \frac{\alpha}{2} - N_z^2)(1 - \alpha - N_x^2) - \frac{\alpha}{2}N_x^2 - N_x^2N_z^2}{1 - \alpha - N_x^2} \\ &= -iE_y\frac{2\sqrt{2}\Delta}{\alpha RZ}\frac{(1 - \frac{\alpha}{2} - N_z^2)(1 - \alpha) - N_x^2 + \frac{\alpha}{2}N_x^2 + N_x^2N_z^2 - \frac{\alpha}{2}N_x^2 - N_x^2N_z^2}{1 - \alpha - N_x^2} \\ &= -iE_y\frac{2\sqrt{2}\Delta}{\alpha RZ}\frac{(1 - \frac{\alpha}{2} - N_z^2)(1 - \alpha) - N_x^2}{1 - \alpha - N_x^2} \\ \left|E^- + \frac{x}{R}\frac{N_x}{N_z}E_z\right|^2 &= \frac{8\Delta^2}{\alpha^2R^2}\left(\frac{(1 - \frac{\alpha}{2} - N_z^2)(1 - \alpha) - N_x^2}{1 - \alpha - N_x^2}\right)^2\frac{|E_y|^2}{|z|^2}\end{aligned}$$

Thus, the perpendicular energy absorption

$$\begin{aligned}\frac{d^2W_\perp}{dtdV} &\simeq \frac{\epsilon_0\omega_p^2}{2}\frac{|\Omega|}{\omega^2}\frac{R}{\Delta}\frac{1}{\sqrt{2\pi}}\frac{8\Delta^2}{\alpha^2R^2}\left(\frac{(1-\frac{\alpha}{2}-N_z^2)(1-\alpha)-N_x^2}{1-\alpha-N_x^2}\right)^2\frac{|E_y|^2}{|Z|^2}e^{-\frac{x^2}{2\Delta^2}} \\ &\simeq \epsilon_0\frac{2\sqrt{2}\omega}{\pi\alpha}\frac{\Delta}{R}|E_y|^2\left(\frac{(1-\frac{\alpha}{2}-N_z^2)(1-\alpha)-N_x^2}{1-\alpha-N_x^2}\right)^2(\sqrt{\pi}\frac{e^{-\frac{x^2}{2\Delta^2}}}{|Z|^2})\end{aligned}$$

Through Poynting's theorem

$$\vec{\nabla}\cdot\vec{S}+\frac{d^2W_\perp}{dtdV}=0$$

$$\vec{S}\sim e^{2i(k_xx-\omega t)}$$

$$\frac{\partial S_x}{\partial x}=2ik_xS_x=-\frac{d^2W_\perp}{dtdV}$$

$$\Rightarrow 2ik_x=-\frac{1}{S_x}\frac{d^2W_\perp}{dtdV}\Rightarrow \text{Real}$$

since $k_x=k_R+ik_I=\text{Re}(k_x)+i\text{Im}(k_x)$

$$\Rightarrow 2\text{Im}(k_x)=\frac{1}{S_x}\frac{d^2W_\perp}{dtdV}$$

But,

$$S_x=\frac{1}{4\mu_0}\text{Re}(\vec{E}\times\vec{B}^*+\frac{1}{2}\vec{E}^*\cdot\frac{\partial}{\partial\vec{N}}\overleftrightarrow{\epsilon}\cdot\vec{E})_x$$

where $\overleftrightarrow{\epsilon}$ is the dielectric tensor in Hot Plasma.

$$S_x=\frac{1}{4\mu_0}\text{Re}[(E_yB_z^*-E_zB_y^*)+\frac{1}{2}(E_x^*X_{xx}+E_y^*X_{yx}+E_z^*X_{zx})]$$

$$\overleftrightarrow{X}=\frac{\partial}{\partial\vec{n}}\overleftrightarrow{\epsilon}\cdot\vec{E}=\hat{x}\frac{\partial}{\partial N_x}[\hat{x}U+\hat{y}V+\hat{z}W]+\hat{y}\frac{\partial}{\partial N_y}[\hat{x}U+\hat{y}V+\hat{z}W]+\hat{z}\frac{\partial}{\partial N_z}[\hat{x}U+\hat{y}V+\hat{z}W]$$

where

$$U=\epsilon_{xx}E_x+\epsilon_{xy}E_y+\epsilon_{xz}E_z$$

$$V=\epsilon_{yx}E_x+\epsilon_{yy}E_y+\epsilon_{yz}E_z$$

$$W=\epsilon_{zx}E_x+\epsilon_{zy}E_y+\epsilon_{zz}E_z$$

since $N_y=0$

$$\overleftrightarrow{X}=\hat{x}\frac{\partial}{\partial N_x}[\hat{x}U+\hat{y}V+\hat{z}W]+\hat{z}\frac{\partial}{\partial N_z}[\hat{x}U+\hat{y}V+\hat{z}W]$$

For S_x, we just calculate below components:

$$
\begin{aligned}
X_{xx} &= \frac{\partial}{\partial N_x} U \\
X_{yx} &= 0 \\
X_{zx} &= \frac{\partial}{\partial N_x} U
\end{aligned}
$$

$$
\begin{aligned}
U &= \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z \\
&= (1 - \frac{\alpha}{4} + \frac{\alpha}{2}\frac{R}{\sqrt{2}\Delta}Z)E_x - i(\frac{\alpha}{4} + \frac{\alpha R}{2\sqrt{2}\Delta}Z)E_y + \frac{\alpha}{4}\frac{N_x}{N_z}2(1 + \frac{x}{\sqrt{2}\Delta}Z)E_z \\
&= (1 - \frac{\alpha}{4})E_x - i\frac{\alpha}{R}E_y + \frac{\alpha}{R}2\sqrt{2}\Delta Z(E_x - iE_y) + \frac{\alpha}{2}\frac{N_x}{N_z}(1 + \frac{x}{\sqrt{2}\Delta}Z)E_z \\
& \quad (\text{put, } E_x - iE_y = E^-) \\
&= (1 - \frac{\alpha}{4})E_x - i\frac{\alpha}{R}E_y + \frac{\alpha R}{2\sqrt{2}\Delta}Z(E^- + \frac{x}{R}\frac{N_x}{N_z}E_z) + \frac{\alpha}{2}\frac{N_x}{N_z}E_z \\
&= (1 - \frac{\alpha}{4})E_x - i\frac{\alpha}{R}E_y + \frac{\alpha}{2}\frac{N_x}{N_z}E_z + \frac{\alpha R}{2\sqrt{2}\Delta}Z(-iE_y)\frac{2\sqrt{2}\Delta}{\alpha R} \times \frac{(1 - \frac{\alpha}{2} - N_z^2)(1-\alpha) - N_x^2}{1 - \alpha - N_x^2} \\
&= \left[(1 - \frac{\alpha}{4})E_x - i\frac{\alpha}{R}E_y + \frac{\alpha}{2}\frac{N_x}{N_z}E_z\right] - iE_y\left[\frac{(1 - \frac{\alpha}{2} - N_z^2)(1-\alpha) - N_x^2}{1 - \alpha - N_x^2}\right]
\end{aligned}
$$

But,

$$
E_z = -\frac{(\epsilon_{xz} + N_xN_z)E^- + iN_xN_zE_y}{\epsilon_{zz} - N_z^2} \simeq -\frac{iN_xN_zE_y}{1 - \alpha - N_x^2}
$$

$$
\therefore \frac{\alpha}{2}\frac{N_x}{N_z}E_z = -iE_y\frac{\alpha N_x^2}{2(1 - \alpha - N_x^2)}
$$

and,

$$
E^- = E_x - iE_y \simeq 0
$$

$$
\therefore E_x \simeq iE_y
$$

The first term in the square bracket in above equation becomes

$$
\begin{aligned}
(1 - \frac{\alpha}{4})E_x - i\frac{\alpha}{4}E_y + \frac{\alpha}{2}\frac{N_x}{N_z}E_z &= (1 - \frac{\alpha}{4})(iE_y) - iE_y\frac{\alpha}{4} - iE_y\frac{\alpha N_x^2}{2(1 - \alpha - N_x^2)} \\
&= iE_y(1 - \frac{\alpha}{4} - \frac{\alpha}{4} - \frac{\alpha N_x^2}{2(1 - \alpha - N_x^2)}) \\
&= iE_y\left[1 - \frac{\alpha}{2} - \frac{\alpha N_x^2}{2(1 - \alpha - N_x^2)}\right]
\end{aligned}
$$

Thus,

$$
U = (+iE_y)\left[1 - \frac{\alpha}{2} - \frac{\alpha N_x^2}{2(1 - \alpha - N_x^2)} - \frac{\left(1 - \frac{\alpha}{2} - N_z^2\right)(1-\alpha) - N_x^2}{1 - \alpha - N_x^2}\right]
$$

$$
\begin{aligned}
X_{xx} = \frac{\partial U}{\partial N_x} &= iE_y \left[-\frac{\alpha}{2} \frac{2N_x(1-\alpha-N_x^2) - N_x^2(-2N_x)}{(1-\alpha-N_x^2)^2} \right. \\
&\quad \left. - \frac{-2N_x(1-\alpha-N_x^2) - \left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha) - N_x^2\right](-2N_x)}{(1-\alpha-N_x^2)^2} \right] \\
&= iE_y \left[-\frac{\alpha}{2} \frac{2N_x(1-\alpha)}{(1-\alpha-N_x^2)^2} \right. \\
&\quad \left. + \frac{-2N_x(1-\alpha) - 2N_x\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)}{(1-\alpha-N_x^2)^2} \right] \\
&= iE_y N_x \left[\frac{-\alpha(1-\alpha) + 2(1-\alpha) - 2\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)}{(1-\alpha-N_x^2)^2} \right] \\
&= iE_y N_x \frac{2(1-\alpha)N_z^2}{(1-\alpha-N_x^2)^2} \\
X_{zx} = \frac{\partial U}{\partial N_z} &= iE_y \left(\frac{2N_z(1-\alpha)}{1-\alpha-N_x^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
E_x^* X_{xx} &\simeq (iE_y)^* X_{xx} \\
&= -iE_y^*(iE_y) N_x \frac{2(1-\alpha)N_z^2}{(1-\alpha-N_x^2)^2} \\
&= |E_y|^2 N_x \frac{2(1-\alpha)N_z^2}{(1-\alpha-N_x^2)^2} \\
E_z^* X_{zx} &\simeq -\frac{N_x N_z (iE_y)^*}{1-\alpha-N_x^2} \cdot iE_y \frac{2N_z(1-\alpha)}{1-\alpha-N_x^2} \\
&= -\frac{N_x N_z (-iE_y^*)}{1-\alpha-N_x^2} \cdot iE_y \frac{2N_z(1-\alpha)}{1-\alpha-N_x^2} \\
&= -|E_y|^2 N_x \frac{2N_z^2(1-\alpha)}{(1-\alpha-N_x^2)^2}
\end{aligned}
$$

$$
\therefore\; E_x^* X_{xx} + E_z^* X_{zx} = 0
$$

And,

$$
\begin{aligned}
& E_y B_z^* - E_z B_y^* \\
= \; & \frac{1}{c}[E_y(N_x E_y^*) - E_z(N_z E_x^* - N_x E_z^*)] \\
= \; & \frac{1}{c} N_x |E_y|^2 - \frac{1}{c}(-iE_y) \frac{N_x N_z}{1-\alpha-N_x^2} \left(N_z(-iE_y^*) + N_x \frac{-iE_y^* N_x N_z}{1-\alpha-N_x^2} \right) \\
= \; & \frac{1}{c} N_x |E_y|^2 + \frac{1}{c}(iE_y)(-iE_y^*) \frac{N_x N_z}{1-\alpha-N_x^2} \left(N_z + \frac{N_x^2 N_z}{1-\alpha-N_x^2} \right) \\
= \; & \frac{N_x}{c}|E_y|^2 \left[1 + \frac{N_z}{1-\alpha-N_x^2} \left(N_z + \frac{N_x^2 N_z}{1-\alpha-N_x^2} \right) \right] \\
= \; & \frac{N_x}{c}|E_y|^2 \left[1 + \frac{N_z}{1-\alpha-N_x^2} \frac{(1-\alpha)N_z - N_x^2 N_z + N_x^2 N_z}{1-\alpha-N_x^2} \right] \\
= \; & \frac{N_x}{c}|E_y|^2 \left[1 + \frac{(1-\alpha)N_z^2}{(1-\alpha-N_x^2)^2} \right]
\end{aligned}
$$

Thus,

$$S_x = \frac{1}{4\mu_0 c} N_x |E_y|^2 \left[1 + \frac{(1-\alpha)N_z^2}{(1-\alpha-N_x^2)} \right]$$

$$\begin{aligned}
2\,\mathrm{Im}(k_x) &= \frac{1}{S_x}\frac{d^2 W_\perp}{dt\,dV} \\
&= 4\mu_0 c \frac{1}{N_x|E_y|^2} \frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha) - N_x^2\right]^2}{(1-\alpha-N_x^2)^2 + (1-\alpha)N_z^2} \\
&\quad \times \epsilon_0 \frac{2\sqrt{2}\omega}{\pi\alpha}\frac{\sqrt{2}\Delta}{R}|E_y|^2 \cdot \sqrt{\pi}\frac{e^{-x^2/2\Delta^2}}{|Z|^2} \\
&= -\frac{8\omega}{c\pi}\frac{\sqrt{2}\Delta}{R}\frac{1}{\alpha N_x}\mathrm{Im}\left(\frac{1}{Z}\right) \frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha) - N_x^2\right]^2}{(1-\alpha-N_x^2)^2 + (1-\alpha)N_z^2}
\end{aligned}$$

where

$$\mathrm{Im}\left(\frac{1}{Z}\right) = \frac{\sqrt{\pi}e^{-x^2/2\Delta^2}}{|Z|^2}.$$

$$\eta = \int_{-\infty}^{\infty} \mathrm{Im}(2k_x)\,dx$$

$f = 1 - e^{-\eta} = \dfrac{W_{abs}}{W_0}$: Fraction of absorbed energy to the input wave energy.

<u>Calculation of</u>

$$\int_{-\infty}^{\infty} \mathrm{Im}\left(\frac{1}{Z}\right)\,dx = \int_{-\infty}^{\infty} \frac{\sqrt{\pi}e^{-x^2/2\Delta^2}}{\left|Z\left(\frac{x}{\sqrt{2}\Delta}\right)\right|^2}dx$$

since $\frac{x}{\sqrt{2}\Delta}$ is real,

$$\begin{aligned}
Z\left(\frac{x}{\sqrt{2}\Delta}\right) &= i\sqrt{\pi}e^{-x^2/2\Delta^2} - 2\frac{x}{\sqrt{2}\Delta}Y\left(\frac{x}{\sqrt{2}\Delta}\right) \\
\mathrm{Im}\left(\frac{1}{Z}\right) &= \mathrm{Im}\left[\frac{\mathrm{Re}Z - i\mathrm{Im}Z}{|Z|^2}\right] = -\frac{\mathrm{Im}Z}{|Z|^2} \\
&= \frac{-\sqrt{\pi}e^{-x^2/2\Delta^2}}{|Z|^2}
\end{aligned}$$

For $\frac{x}{\sqrt{2}\Delta} \gg 1$

$$|Z|^2 \simeq \frac{1}{\left(\frac{x}{\sqrt{2}\Delta}\right)} \quad \text{from Asymptotic Expansion}$$

Thus,

$$\mathrm{Im}\left(\frac{1}{Z}\right) = -\frac{\sqrt{\pi}}{2\Delta^2}x^2 e^{-x^2/2\Delta^2}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{Im}\left(\frac{1}{Z}\right) dx &= -\frac{\sqrt{\pi}}{2\Delta^2}\int_{-\infty}^{\infty} x^2 e^{-x^2/2\Delta^2}\, dx \\
&= -\frac{\sqrt{\pi}}{2\Delta^2}\frac{1}{2}\sqrt{\frac{\pi}{\left(\frac{1}{2\Delta^2}\right)^3}} \\
&= -\frac{\pi\Delta}{\sqrt{2}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\eta &= -\frac{8\omega}{c\pi}\frac{\sqrt{2}\Delta}{R}\frac{1}{\alpha N_x}\left(-\frac{\pi\Delta}{\sqrt{2}}\right)\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-N_x^2\right]^2}{(1-\alpha-N_x^2)^2+(1-\alpha)N_z^2} \\
&= \frac{8}{c\pi}2\pi\frac{c}{\lambda}\cdot\frac{\pi\Delta^2}{R\alpha N_x}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-N_x^2\right]^2}{(1-\alpha-N_x^2)^2+(1-\alpha)N_z^2} \\
&\quad \left(\Delta^2=\frac{k_z^2R^2T}{m\omega^2}\right) \\
&= \frac{8}{c\pi}2\pi\frac{c}{\lambda}\cdot\frac{\pi}{R\alpha N_x}\frac{k_z^2R^2T}{m\omega^2}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-N_x^2\right]^2}{(1-\alpha-N_x^2)^2+(1-\alpha)N_z^2} \\
&= \frac{16\pi}{\alpha}\cdot\frac{R}{\lambda}\frac{N_z^2}{N_x}\frac{T}{mc^2}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-N_x^2\right]^2}{(1-\alpha-N_x^2)^2+(1-\alpha)N_z^2}
\end{aligned}
$$

6.2.3 O-mode & X-mode Heating

A. Fundamental Harmonic ($n=1$)

$$
\eta=\frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{N_x}\frac{T}{mc^2}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-N_x^2\right]^2}{(1-\alpha-N_x^2)^2+(1-\alpha)N_z^2}
$$

where $N_x^2=\frac{1}{2}\left(3-N_z^2-2\alpha\pm\sqrt{(1+N_z^2)^2-4\alpha N_z^2}\right)$ from section 6.1

ⓐ X-mode

$$
N_x^2=\frac{1}{2}\left(3-N_z^2-2\alpha+\sqrt{(1+N_z^2)^2-4\alpha N_z^2}\right)
$$

ⓑ O-mode

$$
N_x^2=\frac{1}{2}\left(3-N_z^2-2\alpha-\sqrt{(1+N_z^2)^2-4\alpha N_z^2}\right)
$$

a. "Near-normal incidence" (N_z is small)

$$
\begin{aligned}
N_x^2 &= \frac{1}{2}\left(3-N_z^2-2\alpha\pm\sqrt{1+2N_z^2+N_z^4-4\alpha N_z^2}\right) \\
&\simeq \frac{1}{2}\left(3-N_z^2-2\alpha\pm\sqrt{1+(2-4\alpha)N_z^2}\right) \\
&\simeq \frac{1}{2}\left(3-N_z^2-2\alpha\pm\left(1+(1-2\alpha)N_z^2-\frac{1}{8}(1-2\alpha)^2N_z^4\right)\right) \\
&\simeq \frac{1}{2}\left(3-N_z^2-2\alpha\pm\left(1+(1-2\alpha)N_z^2\right)\right)
\end{aligned}
$$

i) X-mode

$$
\begin{aligned}
N_x^2 &\simeq \frac{1}{2}\left[3 - N_z^2 - 2\alpha + \left(1 + (1-2\alpha)N_z^2\right)\right] \\
&= \frac{1}{2}\left[3 - N_z^2 - 2\alpha + 1 + N_z^2 - 2\alpha N_z^2\right] \\
&= \frac{1}{2}\left[4 - 2\alpha\left(1 + N_z^2\right)\right] \\
&= 2 - \alpha\left(1 + N_z^2\right) \\
&\simeq 2 - \alpha
\end{aligned}
$$

$$
\begin{aligned}
\eta_X &= \frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{2-\alpha}}\frac{T}{mc^2}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-2+\alpha\right]^2}{\left[1-\alpha-2+\alpha\right]^2+(1-\alpha)N_z^2} \\
&\simeq \frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{2-\alpha}}\frac{T}{mc^2}\frac{\frac{1}{4}[(2-\alpha)(1-\alpha)-2(2-\alpha)]^2}{1} \\
&= 4\pi\frac{R}{\lambda}\frac{T}{mc^2}\cdot N_z^2\frac{(2-\alpha)^{3/2}(1+\alpha)^2}{\alpha}
\end{aligned}
$$

ii) O-mode

$$
\begin{aligned}
N_x^2 &\simeq \frac{1}{2}\left[3 - N_z^2 - 2\alpha - \left(1 + (1-2\alpha)N_z^2\right)\right] \\
&= \frac{1}{2}\left[3 - N_z^2 - 2\alpha - 1 - (1-2\alpha)N_z^2\right] \\
&= \frac{1}{2}\left[2 - 2\alpha - 2N_z^2 + 2\alpha N_z^2\right] \\
&= (1-\alpha) - N_z^2(1-\alpha) \\
&= (1-\alpha)(1-N_z^2) \\
&\simeq 1 - \alpha
\end{aligned}
$$

$$
\begin{aligned}
\eta_O &= \frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{1-\alpha}}\frac{T}{mc^2}\frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-1+\alpha\right]^2}{\left[1-\alpha-1+\alpha\right]^2+(1-\alpha)N_z^2} \\
&\simeq \frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{1-\alpha}}\frac{T}{mc^2}\frac{\frac{1}{4}[(2-\alpha-2N_z^2)(1-\alpha)-2(1-\alpha)]^2}{(1-\alpha)N_z^2} \\
&= \frac{16\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{1-\alpha}}\frac{T}{mc^2}\frac{1}{4}\frac{[(1-\alpha)(2-\alpha-2N_z^2-2)]^2}{(1-\alpha)N_z^2} \\
&= \frac{4\pi}{\alpha}\frac{R}{\lambda}\frac{N_z^2}{\sqrt{1-\alpha}}\frac{T}{mc^2}\frac{(1-\alpha)^2(\alpha+2N_z^2)^2}{(1-\alpha)N_z^2} \\
&\simeq 4\pi\frac{R}{\lambda}\frac{T}{mc^2}\alpha(1-\alpha)^{1/2}
\end{aligned}
$$

or,

$$\begin{aligned} N_x^2 &\simeq (1-\alpha)(1-N_z^2) \\ N_x^2 &= \sqrt{1-\alpha}\sqrt{1-N_z^2} \\ \text{and, } \frac{1}{N_x} &= \frac{1}{\sqrt{(1-\alpha)(1-N_z^2)}} = \frac{1}{\sqrt{1-\alpha-(1-\alpha)N_z^2}} \\ &\simeq \frac{1}{\sqrt{1-\alpha}} \end{aligned}$$

We use $\sqrt{1-\alpha}\sqrt{1-N_z^2}$ (keep N_z term) for N_x^2. For $1/N_x$, we let $N_z \to 0$.

Therefore,

$$\begin{aligned} & \frac{\left[\left(1-\frac{\alpha}{2}-N_z^2\right)(1-\alpha)-(1-\alpha)(1-N_z^2)\right]^2}{[1-\alpha-(1-\alpha)(1-N_z^2)]^2+(1-\alpha)N_z^2} \\ =& \frac{\frac{1}{4}\left[(2-\alpha-2N_z^2)(1-\alpha)-2(1-\alpha)(1-N_z^2)\right]^2}{(1-\alpha)N_z^2((1-\alpha)N_z^2+1)} \\ =& \frac{1}{4}\frac{[(1-\alpha)(2-\alpha-2N_z^2-2+2N_z^2)]^2}{(1-\alpha)N_z^2((1-\alpha)N_z^2+1)} \\ =& \frac{1}{4}\frac{[(1-\alpha)(2-\alpha-2N_z^2-2+2N_z^2)]^2}{(1-\alpha)N_z^2(1+(1-\alpha)N_z^2)} \\ =& \frac{1}{4}\frac{(1-\alpha)^2(-\alpha)^2}{(1-\alpha)N_z^2(1+N_z^2(1-\alpha))} \end{aligned}$$

Thus,

$$\eta_O \simeq 4\pi\frac{R}{\lambda}\frac{T}{mc^2}\frac{\alpha(1-\alpha)^{1/2}}{1+{N_z}^2(1-\alpha)}$$

b. “Normal-incidence” ($N_z = 0$)

i) O-mode

$$\eta_O = 4\pi\frac{R}{\lambda}\frac{T}{mc^2}\frac{\alpha(1-\alpha)^{1/2}}{1+{N_z}^2(1-\alpha)}$$

$$\therefore \eta_O(90) \simeq 4\pi\frac{R}{\lambda}\frac{T}{mc^2}\alpha(1-\alpha)^{1/2}$$

ii) X-mode

$$\eta_X = 4\pi\frac{R}{\lambda}\frac{T}{mc^2}{N_z}^2\frac{(2-\alpha)^{3/2}(1+\alpha)^2}{\alpha}$$

$$\therefore \eta_X(90) = 0$$

⇒ “Expanding to the next-order in the temperature is necessary”

$$\eta_X(90) \simeq \frac{\pi}{2}\frac{R}{\lambda}\frac{T^2}{m^2c^4}\alpha(2-\alpha)^{3/2}$$

B. Second Harmonic ($n = 2$)

$$\eta = 2\pi \frac{R}{\lambda} \frac{T}{mc^2} (\frac{{N_x}^2 n^2 T}{2mc^2})^{n-2} \frac{n^3}{(n-1)!} \alpha N_x \frac{(S-D-N^2)^2 (P-N_x^2)^2}{D^2(P-N_x^2)^2 + PN_z^2(S-N^2)^2}$$

$$\eta(n=2) = 2\pi \frac{R}{\lambda} \frac{T}{mc^2} 8\alpha N_x \frac{(S-D-N^2)^2 (P-N_x^2)^2}{D^2(P-N_x^2)^2 + PN_z^2(S-N^2)^2}$$

where

$$S = 1 - \frac{{\omega_p}^2}{\omega^2 - \Omega^2} = 1 - \frac{{\omega_p}^2}{\omega^2(1-\frac{\Omega^2}{\omega^2})} = 1 - \frac{\alpha}{1-\frac{1}{4}} = 1 - \frac{4}{3}\alpha$$

$$P = 1 - \frac{{\omega_p}^2}{\omega^2} = 1 - \alpha$$

$$D = -\frac{{\omega_p}^2}{\omega^2 - \Omega^2} \frac{\Omega}{\omega} = -\frac{4}{3}\alpha \frac{1}{2} = -\frac{2}{3}\alpha$$

$${N_x}^2 = \frac{1}{2A}[-B \pm \sqrt{B^2 - 4AC}]$$

$$A = S$$

$$B = -(S+P)(S-{N_z}^2) + D^2$$

$$C = P[(S-{N_z}^2)^2 - D^2]$$

(+) sign : O-mode
(−) sign : X-mode

$$\begin{aligned} B^2 - 4AC &= [(S+P)(S-{N_z}^2) - D^2]^2 - 4SP[(S-{N_z}^2)^2 - D^2] \\ &= [(S-P)(S-{N_z}^2) - D^2]^2 \end{aligned}$$

- O-mode:

$$\begin{aligned} {N_x}^2 &= \frac{1}{2S}[(S+P)(S-{N_z}^2) - D^2 + (S-P)(S-{N_z}^2) - D^2] \\ &= \frac{1}{2S}[(S-{N_z}^2)(S+P+S-P) - 2D^2] \\ &= \frac{1}{S}[S(S-{N_z}^2) - D^2] \end{aligned}$$

- X-mode:

$$\begin{aligned} {N_x}^2 &= \frac{1}{2S}[(S+P)(S-{N_z}^2) - D^2 - (S-P)(S-{N_z}^2) + D^2] \\ &= \frac{1}{S}(S-{N_z}^2)P \end{aligned}$$

◇ Normal incidence ($N_z = 0$)

- O-mode

$$\begin{aligned}
N_x^2 &= \frac{1}{S}(S^2 - D^2) = \frac{1}{S}(S+D)(S-D) \\
&= 3\frac{1}{3-4\alpha}(1 - \frac{4}{3}\alpha - \frac{2}{3}\alpha)(1 - \frac{4}{3}\alpha + \frac{2}{3}\alpha) \\
&= \frac{1}{3-4\alpha}(1-2\alpha)(3-2\alpha)
\end{aligned}$$

$$N_x = \frac{1}{(3-4\alpha)^{1/2}}(1-2\alpha)^{1/2}(3-2\alpha)^{1/2}$$

$$\eta_O(90^\circ, n=2) = 2\pi\frac{R}{\lambda}\frac{T}{mc^2}8\alpha N_x\frac{(S-D-N_x^2)^2}{D^2}$$

$$\begin{aligned}
S - D - N_x^2 &= (1 - \frac{4}{3}\alpha + \frac{2}{3}\alpha) - \frac{1}{3-4\alpha}(1-2\alpha)(3-2\alpha) \\
&= (1 - \frac{2}{3}\alpha) - \frac{1}{3-4\alpha}(1-2\alpha)(3-2\alpha) \\
&= \frac{1}{3}(3-2\alpha) - \frac{(1-2\alpha)(3-2\alpha)}{3-4\alpha} \\
&= \frac{(3-2\alpha)(3-4\alpha)) - 3(1-2\alpha)(3-2\alpha)}{3(3-4\alpha)} \\
&= \frac{(3-2\alpha)(3-4\alpha-3+6\alpha)}{3(3-4\alpha)} \\
&= \frac{2}{3}\alpha\frac{3-2\alpha}{3-4\alpha}
\end{aligned}$$

$$\begin{aligned}
\eta_O(90^\circ, n=2) &= 2\pi\frac{R}{\lambda}\frac{T}{mc^2}8\alpha\frac{(1-2\alpha)^{1/2}(3-2\alpha)^{5/2}}{(3-4\alpha)^{5/2}(-\frac{2}{3}\alpha)^2}\frac{4}{9}\alpha \\
&= 16\pi\frac{R}{\lambda}\frac{T}{mc^2}\alpha\frac{(1-2\alpha)^{1/2}(3-2\alpha)^{5/2}}{(3-4\alpha)^{5/2}}
\end{aligned}$$

- X-mode

$$N_x^2 = \frac{1}{S}(S - N_z^2)P = P = 1 - \alpha$$

$$N_x = (1-\alpha)^{1/2}$$

$$\eta_X(90^\circ, n=2) = 2\pi\frac{R}{\lambda}\frac{T}{mc^2}8\alpha N_x\frac{(S-D-N_x^2)^2}{D^2}$$

$$\begin{aligned}
S - D - N_x^2 &= S - D - P \\
&= 1 - \frac{4}{3}\alpha + \frac{2}{3}\alpha - 1 + \alpha \\
&= \frac{\alpha}{3}
\end{aligned}$$

$$
\begin{aligned}
\eta_X(90^\circ, n=2) &= 16\pi \frac{R}{\lambda} \frac{T}{mc^2} \alpha (1-\alpha)^{1/2} \frac{(\frac{\alpha}{3})^2}{(-\frac{2}{3}\alpha)^2} \\
&= 4\pi \frac{R}{\lambda} \frac{T}{mc^2} \alpha (1-\alpha)^{1/2}
\end{aligned}
$$