5 Landau Damping

In the first order, we denote the perturbation in $f(\vec{r}, \vec{v}, t)$ by $f_1(\vec{r}, \vec{v}, t)$:

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t)$$

The first-order Vlasov equation for electron is

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla} f_1 - \frac{e}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}} = 0$$

Where we let $\vec{B}_0 = \vec{E}_0 = 0$
and we assumed the ions are massive and fixed
and that the waves are plane waves in the x direction

$$f_1 \propto e^{i(kx-\omega t)}$$

Then the first-order Vlasov equation becomes

$$-i\omega f_1 + ikv_x f_1 = \frac{e}{m} E_x \frac{\partial f_0}{\partial v_x}$$
$$\therefore f_1 = \frac{ieE_x}{m} \frac{\partial f_0 / \partial v_x}{\omega - kv_x}$$

Poisson's equation

$$\epsilon_0 \vec{\nabla} \cdot \vec{E}_1 = ik\epsilon_0 E_x = -en_1 = -e \int\int\int f_1 d^3v$$
$$ik\epsilon_0 E_x = -e \int\int\int \frac{ieE_x}{m} \frac{\partial f_0/\partial v_x}{\omega - kv_x} d^3v$$
$$\rightarrow 1 = \frac{-e^2}{km\epsilon_0} \int\int\int \frac{\partial f_0/\partial v_x}{\omega - kv_x} d^3v$$

If we replace f_0 by a normalized function $\hat{f}_0$;
that is, $f_0 = n_0 \hat{f}_0$

$$1 = -\frac{\omega_p^2}{k} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} \frac{\partial \hat{f}_0(v_x, v_y, v_z)/\partial v_x}{\omega - kv_x} dv_x$$

For a one-dimensional Maxwellian distribution

$$1 = -\frac{\omega_p^2}{k} \int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - kv_x} dv_x$$
$$= \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v_x}{v_x - \omega/k} dv_x$$

Dropping the subindex x,

$$1 = \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial \hat{f}_0/\partial v}{v-\omega/k}dv \quad : \text{dispersion relation}$$

singularity at $v = \omega/k$
No problem, because in practice ω is almost never real.
The integral must be treated as a contour integral in the complex v plane.

- For an unstable wave, with $\mathrm{Im}(\omega) > 0$

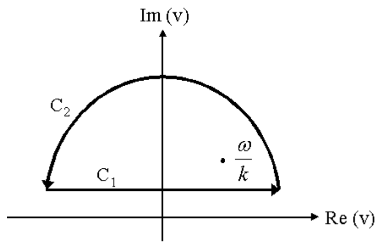

Figure 2: Contour I

- For a damped wave, with $\mathrm{Im}(\omega) < 0$

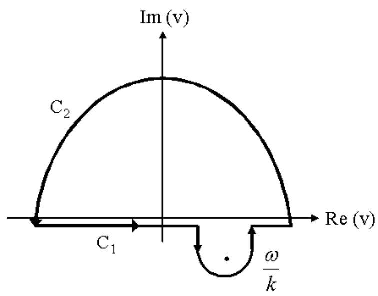

Figure 3: Contour II

$$\int_{C_1} G dv + \int_{C_2} G dv = 2\pi i R(\omega/k)$$

Where G is the integrand, C_1 is the path along the real axis, C_2 is the semicircle at infinity, and $R(\omega/k)$ is the residue at ω/k.
This Works If The Integral over C_2 Vanishes. Unfortunately, this does not happen for a Maxwellian Distribution, which contains the factor

$$\exp(-v^2/v_{th}^2)$$

This factor becomes large for $v \to \pm i\infty$, and the contribution from C_2 cannot be neglected.

But, for the case of large phase velocity and weak damping (small imaginary Im(ω), the contour integral is possible as shown in Fig. 5.

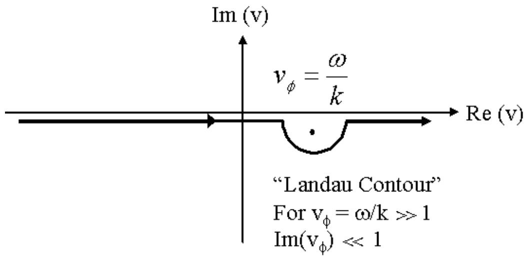

Figure 4: Contour III

- An approximate dispersion relation for the case of large phase velocity and weak damping.

  For $\mathrm{Re}(v_\phi) \gg 1$ and $\mathrm{Im}(v_\phi) \ll 1$, the contour in Fig. 5 is used. Then the dispersion relation becomes

$$1 = \frac{\omega_p^2}{k^2}\left[\ P\int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v}{v-(\omega/k)} dv + i\pi \frac{\partial \hat{f}_0}{\partial v}\Big|_{v=\frac{\omega}{k}}\ \right]$$

  Where P stands for the Cauchy principal value.

  1) The evaluation of $P\int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v}{v-(\omega/k)} dv$: stop just before encountering the pole

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v}{v-(\omega/k)} dv &= \left[\ \frac{\hat{f}_0}{v-v_\phi}\ \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-\hat{f}_0}{(v-v_\phi)^2} dv \\
&= \int \frac{\hat{f}_0}{(v-v_\phi)^2} dv \quad (\because \hat{f}_0 << 1 \ \text{ for large in } v_\phi) \\
&= \overline{(v-v_\phi)^{-2}}
\end{aligned}$$

  Since $v_\phi >> v$

$$\begin{aligned}
(v-v_\phi)^{-2} &= v_\phi^{-2}\left(1-\frac{v}{v_\phi}\right)^{-2} \\
&= v_\phi^{-2}\left(1+\frac{2v}{v_\phi}+\frac{3v^2}{v_\phi^2}+\frac{4v^3}{v_\phi^3}+\cdots\right)
\end{aligned}$$

  The odd terms vanish upon taking the average,

$$\overline{(v-v_\phi)^{-2}} \simeq v_\phi^{-2}\left(1+\frac{3\overline{v^2}}{v_\phi^2}\right)$$

$$\frac{1}{2}m\overline{v^2} = \frac{1}{2}kT_e$$

Thus, the dispersion relation becomes

$$
\begin{aligned}
1 &= \frac{\omega_p^2}{k^2}\frac{1}{v_p^2}(1+3\frac{1}{v_p^2}\frac{kT_e}{m}) \\
&= \frac{\omega_p^2}{k^2}\frac{k^2}{\omega^2}(1+3\frac{k^2}{\omega^2}\frac{kT_e}{m}) \\
\therefore \omega^2 &= \omega_p^2 + \frac{\omega_p^2}{\omega^2}\frac{3kT_e}{m}k^2
\end{aligned}
$$

We assumed $\mathrm{Im}(\omega/k) << 1$
If the thermal correction is small, $\omega^2 \approx \omega_p^2$

$$
\therefore \omega^2 = \omega_p^2 + \frac{3kT_e}{m}k^2
$$

2) The evaluation of the imaginary term
Neglect the thermal correction to the real part of ω

$$
\omega^2 \simeq \omega_p^2
$$

$$
\begin{aligned}
1 &= \frac{\omega_p^2}{k^2}\left[\; P\int_{-\infty}^{\infty}\frac{\partial \hat{f}_0/\partial v}{v-(\omega/k)}dv + i\pi\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}} \;\right] \\
&= \frac{\omega_p^2}{k^2}\frac{1}{v_\phi^2} + \frac{\omega_p^2}{k^2}i\pi\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}} \\
&= \frac{\omega_p^2}{\omega} + i\pi\frac{\omega_p^2}{k^2}\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}}
\end{aligned}
$$

$$
\omega^2\left(1 - i\pi\frac{\omega_p^2}{k^2}\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}}\right) = \omega_p^2
$$

$$
\therefore \omega = \omega_p\left(1 - i\pi\frac{\omega_p^2}{k^2}\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}}\right)^{-1/2} \simeq \omega_p\left(1 + i\frac{\pi}{2}\frac{\omega_p^2}{k^2}\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}}\right)
$$

$$
\hat{f}_0 = \frac{1}{\sqrt{\pi}v_{th}}\exp(-\frac{v^2}{v_{th}^2}) : \quad \mathrm{one-dimensional\ Maxwellian} \quad v_{th}^2 = \frac{2kT_e}{m}
$$

$$
\begin{aligned}
\frac{\partial \hat{f}_0}{\partial v} &= (\pi v_{th}^2)^{-1/2}(\frac{-2v}{v_{th}^2})\exp(-\frac{v^2}{v_{th}^2}) \\
&= -\frac{2v}{\sqrt{\pi}v_{th}^3}\exp(-\frac{v^2}{v_{th}^2}) \\
\frac{\partial \hat{f}_0}{\partial v}|_{v=\frac{\omega}{k}} &= -\frac{2\frac{\omega}{k}}{\sqrt{\pi}v_{th}^3}\exp(-\frac{\omega^2/k^2}{v_{th}^2})
\end{aligned}
$$

$$\mathrm{Im}(\omega) = -\frac{\pi}{2}\frac{\omega_p^3}{k^2}\frac{2}{\sqrt{\pi}}\frac{\omega}{k}\frac{1}{v_{th}^3}\exp(-\frac{\omega^2/k^2}{v_{th}^2})$$

$$\omega^2 = \omega_p^2 + \frac{3kT_e}{m}k^2$$

keep thermal correction term in the exponent.

$$\begin{aligned}
\mathrm{Im}(\omega) &= -\frac{\pi}{2}\frac{\omega_p^3}{k^2}\frac{2}{\sqrt{\pi}}\frac{\omega}{k}\frac{1}{v_{th}^3}\exp(-\frac{\omega_p^2/k^2}{v_{th}^2})\exp(-\frac{3}{2}) \\
&= -\sqrt{\pi}\omega_p(\frac{\omega_p}{kv_{th}})^3\exp(-\frac{\omega_p^2/k^2}{v_{th}^2})\exp(-\frac{3}{2}) \\
\therefore \mathrm{Im}(\frac{\omega}{\omega_p}) &= -0.22\sqrt{\pi}(\frac{\omega_p}{kv_{th}})^3\exp(-\frac{1}{2k^2\lambda_D^2})
\end{aligned}$$

Where $\lambda_D^2 = \frac{v_{th}^2}{2\omega_p^2}$

If $\mathrm{Im}(\omega) < 0$, collisionless damping of plasma waves: "Landau damping"

This is the analytical result.

- The contour integral by numerical approach was presented (J.D. Jackson, Plasma Phys. 1(1960) pp. 5)
  → Fried and Conte have provided tables for the case when $\hat{f}_0$ is a Maxwellian.

The below figure shows the analytical results and numerical results.

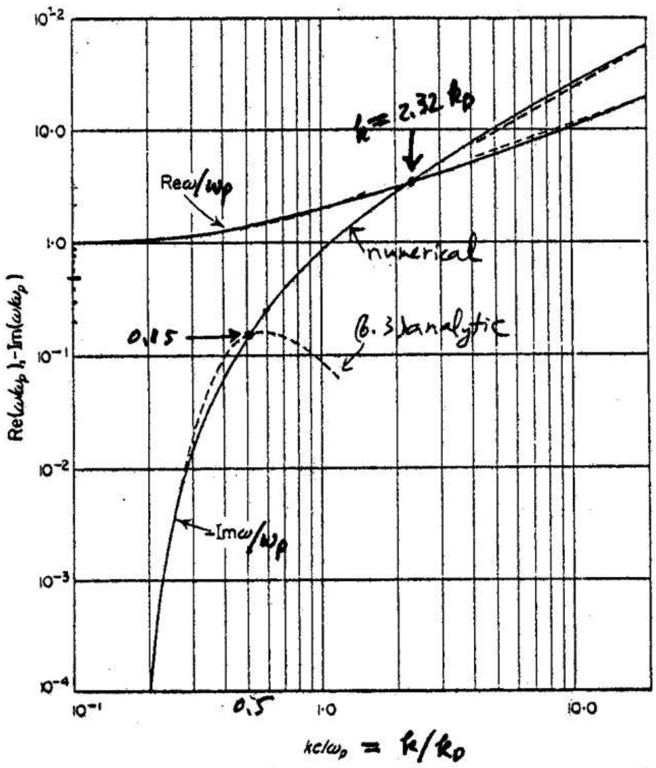

Figure 5: Real and imaginary parts of the frequency as a function of wave number for a stationary one-component plasma in thermal equilibrium. The frequency is given in units of ω_p, while the wave number is expressed in units of the Debye wave number (k_D). The dotted curves represent approximate formulas derived in this section.

- From the Harris Dispersion Relation

  The electrostatic dispersion relation in hot plasma

$$1+\sum_s \frac{1}{k^2\lambda_D^2}\sum_n e^{-b} In(b)[1+\frac{\omega}{k_z v_{th}} Z_n(\zeta_n)] = 0$$

$$\begin{aligned} where\ \lambda_D^2 &= \frac{v_{th}^2}{2\omega_p^2},\quad b = \frac{v_{th}^2 k_\perp^2}{2\Omega^2},\quad \zeta_n = \frac{\omega - n\Omega}{k_z v_{th}} \\ v_{th} &= \sqrt{\frac{2kT_e}{m}} \end{aligned}$$

$Z_n(\zeta_n)$ is "Fried-Conte" function or "Dispersion function"
$Z_n(\zeta_n)$ can be evaluated numerically.

For Landau damping in unmagnetized plasmas ($B_0 = 0$)

$$\begin{aligned} &\Omega \to 0, \qquad k_\perp \to 0, \quad (n \to 0) \\ &k_z \to k \end{aligned}$$

$$\therefore 1 + \frac{1}{k^2\lambda_D^2}[\ 1+\frac{\omega}{kv_{th}} Z(\frac{\omega}{kv_{th}})\ ] = 0$$

⊙ Power Series of $Z(\zeta)$

for $\zeta << 1$,

$$\begin{aligned} Z(\zeta) &= i\pi^{1/2}e^{-\zeta^2} - 2\zeta[1 - 2\zeta^2/3 + 4\zeta^4/15 - 8\zeta^6/105 + \cdots] \\ &= i\pi^{1/2}e^{-\zeta^2} - \zeta\sum_{n=0}^{\infty}(-\zeta^2)^n\pi^{1/2}/(n+1/2)! \end{aligned}$$

⊙ Asympotic Expansion

for $\zeta >> 1$,

$$\begin{aligned} Z(\zeta) &\simeq i\pi^{1/2}\sigma e^{-\zeta^2} - \frac{1}{3}\left[1+\frac{1}{2\zeta^2}+\frac{3}{4\zeta^4}+\cdots\right] \\ &= i\pi^{1/2}\sigma e^{-\zeta^2} - \sum_{n=0}^{\infty}\zeta^{-(2n+1)}(n-\frac{1}{2})!/\pi^{1/2} \end{aligned}$$

$$\text{Where } \sigma = \left\{ \begin{array}{lll} 0 & y>0 & \\ 1 & y=0 & \zeta = x+iy \\ 2 & y<0 & \end{array} \right\}$$

Assuming $v_\phi = \frac{\omega}{k} >> v_{th}$

$$Z(\frac{\omega}{kv_{th}}) \simeq i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} - \frac{kv_{th}}{\omega} - \frac{k^3 v_{th}^3}{2\omega^3} - \frac{3k^5 v_{th}^5}{4\omega^5}$$

A. Real term in dispersion relation

$$\begin{aligned}
1 \quad & + \frac{1}{k^2\lambda_D^2}\left[1 + \frac{\omega}{kv_{th}}\left(-\frac{kv_{th}}{\omega} - \frac{k^3 v_{th}^3}{2\omega^3} - \frac{3k^5 v_{th}^5}{4\omega^5}\right)\right] = 0 \\
\rightarrow \quad & 1 + \frac{1}{k^2\lambda_D^2}\left[1 - 1 - \frac{k^2 v_{th}^2}{2\omega^2} - \frac{3k^4 v_{th}^4}{4\omega^4}\right] = 0 \\
\rightarrow \quad & 1 + \frac{1}{k^2\lambda_D^2}\left[\left(-\frac{k^2 v_{th}^2}{2\omega^2}\right)\left(1 + \frac{3k^2 v_{th}^2}{2\omega^2}\right)\right] = 0 \\
\rightarrow \quad & 1 + \frac{2\omega_p^2}{k^2 v_{th}^2}\left(-\frac{k^2 v_{th}^2}{2\omega^2}\right)\left(1 + \frac{3k^2 v_{th}^2}{2\omega^2}\right) = 0 \\
\rightarrow \quad & 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{th}^2}{2\omega^2}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\therefore \omega^2 &= \omega_p^2 + \frac{\omega_p^2}{\omega^2}\frac{3k^2 v_{th}^2}{2} \\
&= \omega_p^2 + \frac{\omega_p^2}{\omega^2}\left(\frac{3}{2}\frac{2kT_e}{m}k^2\right) \\
&= \boxed{\omega_p^2 + \frac{\omega_p^2}{\omega^2}\frac{3kT_e}{m}k^2}
\end{aligned}$$

B. Imaginary term in dispersion relation

$$\begin{aligned}
0 &= 1 + \frac{1}{k^2\lambda_D^2}\left[1 + \frac{\omega}{kv_{th}}(i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} - \frac{kv_{th}}{\omega} - \frac{k^3 v_{th}^3}{2\omega^3} - \frac{3k^5 v_{th}^5}{4\omega^5})\right] \\
0 &= 1 + \frac{1}{k^2\lambda_D^2}\left[1 + \frac{\omega}{kv_{th}} i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} - 1 - \frac{k^2 v_{th}^2}{2\omega^2}\right] \\
&= 1 + \frac{2\omega_p^2}{k^2 v_{th}^2}\left(\frac{\omega}{kv_{th}}\right) i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} - \frac{2\omega_p^2}{k^2 v_{th}^2}\frac{k^2 v_{th}^2}{2\omega^2} \\
\frac{\omega^2}{\omega_p^2} &= 1 - \frac{2\omega^3}{k^3 v_{th}^3} i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} \\
\therefore \frac{\omega}{\omega_p} &= \left[\quad 1 - 2(\frac{\omega}{kv_{th}})^3 i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} \quad\right]^{1/2} \\
&\simeq 1 - \left(\frac{\omega}{kv_{th}}\right)^3 i\pi^{1/2}\sigma e^{-\frac{\omega^2}{k^2 v_{th}^2}} \\
&\simeq 1 - \left(\frac{\omega_p}{kv_{th}}\right)^3 i\pi^{1/2}\sigma e^{-\frac{\omega_p^2}{k^2 v_{th}^2}} e^{-\frac{3}{2}}
\end{aligned}$$

$$\therefore \boxed{\text{Im}(\frac{\omega}{\omega_p}) = -0.22\sqrt{\pi}\sigma(\frac{\omega_p}{kv_{th}})^3 e^{-\frac{1}{2k^2\lambda_D^2}}}$$