3 Dispersion Relations in a Hot Plasma

3.1 Electromagnetic Dispersion Relation

Vlasov Equation for a Collisionless Plasmas

$$\frac{\partial f_s}{\partial t}(\vec{r},\vec{v},t) + \vec{v}\cdot\vec{\nabla}_r f_s(\vec{r},\vec{v},t) + \left(\frac{q_s}{m_s}\vec{E} + \frac{q_s}{m_s}\vec{v}_s \times \vec{B}\right)\vec{\nabla}_v f_s = 0$$

Maxwell's equations

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\sum_s q_s \int f_s \,\mathrm{d}^3 v \\
&\frac{1}{\mu_0}\,\vec{\nabla}\times\vec{B} = \epsilon_0\frac{\partial\vec{E}}{\partial t} + \sum_s q_s \int \vec{v} f_s \,\mathrm{d}^3 v \\
&\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\
&\vec{\nabla}\cdot\vec{B} = 0
\end{aligned}$$

Let

$$\begin{aligned}
f_s(\vec{r},\vec{v},t) &= f_{so}(\vec{r},\vec{v}) + f_{s1}(\vec{r},\vec{v},t) \\
\vec{B} &= \vec{B}_0(\vec{r}) + \vec{B}_1 \\
\vec{E} &= \vec{E}_0 + \vec{E}_1 = 0 + \vec{E}_1
\end{aligned}$$

and, $f_{s1}, \vec{B}_1, \vec{E}_1$ are dependent of $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$.

① Zeroth order

$$\begin{aligned}
&\frac{\partial f_{s0}}{\partial t} = 0 \\
&\vec{v}\cdot\vec{\nabla}_r f_{s0} + \left(\frac{q_s}{m_s}\vec{v}\times\vec{B}_0\right)\cdot\vec{\nabla}_v f_s = 0 \\
&\vec{\nabla}\cdot\vec{E}_0 = \frac{1}{\epsilon_0}\sum_s q_s \int f_{s0}\,\mathrm{d}^3 v \\
&\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \sum_s q_s \int \vec{v} f_{s0}\,\mathrm{d}^3 v
\end{aligned}$$

② First order

$$\underbrace{\frac{\partial f_{s1}}{\partial t} + \vec{v}_s\cdot\vec{\nabla}_r f_{s1} + \left(\frac{q_s}{m_s}\vec{v}_s\times\vec{B}_0\right)\cdot\vec{\nabla}_v f_{s1}}_{\frac{\mathrm{d}f_{s1}}{\mathrm{d}t}} = \underbrace{-\frac{q_s}{m_s}\left(\vec{E}_1 + \vec{v}_s\times\vec{B}_1\right)\cdot\vec{\nabla}_v f_{s0}}_{S(\vec{r},\vec{v},t)}$$

$$\begin{cases} \mathrm{i}\vec{k}\cdot\vec{E}_1 = \frac{1}{\epsilon_0}\sum_s q_s \int f_{s0}\,\mathrm{d}^3v \\ \frac{1}{\mu_0}\vec{k}\times\vec{B}_1 = -\omega\left(\epsilon_0\vec{E}_1 + \frac{i}{\omega}\sum_s q_s\int \vec{v}f_{s1}\,\mathrm{d}^3v\right) \\ \vec{B}_1 = \frac{1}{\omega}\vec{k}\times\vec{E} \end{cases}$$

Let $\vec{E}_1 + \frac{i}{\epsilon_0\omega}\sum_s q_s \int \vec{v} f_{s1}\,\mathrm{d}^3v = \overleftrightarrow{K}\cdot\vec{E}_1$

$$\begin{aligned} \therefore \frac{1}{\mu_0}\vec{k}\times\vec{B}_1 &= -\omega\epsilon_0 \overleftrightarrow{K}\times\vec{E}_1 = -\omega \overleftrightarrow{\epsilon}\cdot\vec{E}_1 \\ &= \frac{1}{\mu_0\omega}\vec{k}\times(\vec{k}\times\vec{E}_1) \end{aligned}$$

$\therefore \quad \vec{k}\times(\vec{k}\times\vec{E}_1) + \mu_0\epsilon_0\omega^2 \overleftrightarrow{K}\cdot\vec{E}_1 = 0$

$\Rightarrow \quad \vec{k}(\vec{k}\cdot\vec{E}_1) - k^2\vec{E}_1 + \mu_0\omega^2\epsilon_0 \overleftrightarrow{K}\cdot\vec{E}_1 = (\vec{k}\vec{k} + \mu_0\epsilon_0\omega^2 \overleftrightarrow{K} - k^2\overrightarrow{1}) = 0$

Thus Det $(\vec{k}\vec{k} + \mu_0\epsilon_0\omega^2 \overleftrightarrow{K} - k^2 \overleftrightarrow{1}) = 0$

$$F_{s1} = \int_{-\infty}^{t} S\,\mathrm{d}t' = -\frac{q_s}{m_s}\int_{-\infty}^{t}\mathrm{d}t'[\vec{E}_1(\vec{r}'(t'),t') + \vec{v}_s(t')\times\vec{B}_1(\vec{r}'(t'),t')]\cdot\vec{\nabla}_{v'}f_{s0}$$

when

$$\begin{aligned} S &= -\frac{q_s}{m_s}[\vec{E}_1(\vec{r}'(t'),t') + \vec{v}_s(t')\times\vec{B}_1(\vec{r}'(t'),t')]\cdot\vec{\nabla}_{v'}f_{s0} \\ &= -\frac{q_s}{m_s}[\vec{E}_1(\vec{r}'(t'),t') + \vec{v}(t')\times\frac{1}{\omega}(\vec{k}\times\vec{E}_1)]\cdot\vec{\nabla}_{v'}f_{s0} \end{aligned}$$

Since $\vec{v}\times(\vec{k}\times\vec{E}_1) = (\vec{v}\cdot\vec{E}_1)\vec{k} - (\vec{k}\cdot\vec{v})\vec{E}_1$

$$S = -\frac{q_s}{m_s}\left[\left(1 - \frac{\vec{k}\cdot\vec{v}'(t')}{\omega}\right)\vec{E}_1(\vec{r}',t') + \frac{1}{\omega}(\vec{v}'(t')\cdot\vec{E}_1(\vec{r}',t'))\vec{k}\right]\cdot\vec{\nabla}_{v'}f_{s0}$$

Thus

$$f_1(\vec{r},\vec{v},t) = -\frac{q}{m}\int_{-\infty}^{t}\left[\left(1-\frac{\vec{k}\cdot\vec{v}'(t')}{\omega}\right)\vec{E}_1(\vec{r}',t') + \frac{1}{\omega}(\vec{v}'(t')\cdot\vec{E}_1(\vec{r}',t'))\vec{k}\right]\cdot\vec{\nabla}_{v'} f_0\,\mathrm{d}t'$$

Where, I dropped the sub-index s

It is assumed that

$$\vec{E}_1(\vec{r}',t') = \vec{E}\exp[i(\vec{k}\cdot\vec{r}' - \omega t')]$$

$$f_0(\vec{r},\vec{v}) = f(v_\perp, v_z)$$

$$v_\perp^2 = v_x^2 + v_y^2$$

$$\frac{\mathrm{d}\vec{r}'}{\mathrm{d}t} = \vec{v}', \qquad \frac{\mathrm{d}\vec{v}'}{\mathrm{d}t} = \frac{q}{m}\vec{v}'\times\vec{B}_0$$

⊙ Particle motion in a uniform field

$$\begin{aligned}\frac{\partial\vec{v'}}{\partial t'} &= \frac{q}{m}(\vec{E}+\vec{v'}\times\vec{B_0})\\ \vec{v'} &= v'(\vec{t'})\\ \vec{E} &= 0\\ \vec{B_0} &= B_0\hat{z}\end{aligned}$$

Let $\frac{q\vec{B_0}}{m} = \vec{\Omega}$ (sign contained)

$$\begin{aligned}\frac{\partial\vec{v'}}{\partial t'} = \vec{v'}\times\vec{\Omega} &= v_y'\Omega\hat{x} - v_x'\Omega\hat{y} + 0\hat{z}\\ \Rightarrow \frac{\partial\vec{v'_\parallel}}{\partial t'} &= 0\\ \therefore v_\parallel' = v_z'(t') &= \text{const} = v_z(t), t' < t\end{aligned}$$

$$\begin{aligned}\frac{dv_x'}{dt'} &= v_y'\Omega\\ \frac{dv_y'}{dt'} &= -v_x'\Omega\end{aligned}$$

$$(1) + i(2) = \frac{\partial v_x'}{\partial t'} + i\frac{\partial v_y'}{\partial t'} = v_y'\Omega - iv_x'\Omega'$$
$$\Rightarrow \frac{d}{dt'}(v_x' + iv_y') = -i\Omega(v_x' + iv_y')$$

Let $v_x' + iv_y' = v^{+'}$

$$\Rightarrow \frac{d}{dt'}v^{+'} = -i\Omega v^{+'}$$
$$(\frac{d}{dt'}v^{+'} + i\Omega v^{+'} = 0) \times e^{i\Omega t'}$$
$$\Rightarrow \frac{d}{dt'}(v^{+'}e^{i\Omega t'}) = 0$$
$$\int_t^{t'} \frac{d}{dt''}(v^{+'}e^{i\Omega t''})dt'' = v^{+'}(t')e^{i\Omega}t' - v^{+'}(t)e^{i\Omega}t = 0$$

$$\begin{aligned}
\Rightarrow v^{+'}(t') &= v^{+'}(t')e^{i\Omega(t-t')} \\
&= v_\perp e^{i\alpha}e^{i\Omega(t-t')} \\
&= v_\perp e^{i(\alpha+\Omega(t-t'))} \\
&= v_\perp e^{i(\alpha-\Omega(t'-t))} \\
\therefore v'_x(t') &= \mathrm{Re}[v^{+'}(t')] = v_\perp \cos(\alpha - \Omega(t'-t)) \\
v'_y(t') &= \mathrm{Im}[v^{+'}(t')] = v_\perp \sin(\alpha - \Omega(t'-t)) \\
v'_z(t') &= v^{+'}(t) = v_z(t)
\end{aligned}$$

Note Ω contains sign

$$\begin{aligned}
\frac{dx'(t')}{dt'} &= v'_x(t') = v_\perp \cos(\alpha - \Omega(t'-t)) \\
\frac{dy'(t')}{dt'} &= v'_y(t') = v_\perp \sin(\alpha - \Omega(t'-t)) \\
\frac{dz'(t')}{dt'} &= v'_z(t') = v_z
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_t^{t'} \frac{x'(t')}{dt"}dt" = x'(t') - x'(t) &= \frac{v_\perp}{\Omega}[\sin(\alpha - \Omega(t'-t')) - \sin(\alpha - \Omega(t'-t))] \\
&= \frac{v_\perp}{\Omega}[\sin(\alpha - \sin(\alpha - \Omega(t'-t))]
\end{aligned}$$

$$\begin{aligned}
\therefore x'(t') &= x + \frac{v_\perp}{\Omega}[\sin(\alpha) - \sin(\alpha)\cos(\Omega(t'-t)) + \cos(\alpha)\sin\Omega(t'-t)]] \\
&= x + \frac{v_\perp}{\Omega}[\sin(\alpha)(1 - \cos(\Omega(t'-t))) + \cos(\alpha)\sin\Omega(t'-t)]] \\
&= x + \frac{1}{\Omega}[v_{x0}\sin\Omega(t'-t) + v_{y0}(1 - \cos\Omega(t'-t)]
\end{aligned}$$

Similarly

$$\begin{aligned}
y'(t') &= y - \frac{1}{\Omega}[v_{x0}\cos\Omega(t'-t) - v_{y0}(1-\sin\Omega(t'-t)] \\
&= y + \frac{1}{\Omega}[-v_{x0}\cos\Omega(t'-t) + v_{y0}(1-\sin\Omega(t'-t)] \\
z'(t') &= z + v_z(t'-t)
\end{aligned}$$

A. Calculation of

$$[(1 - \frac{\vec{k}\cdot\vec{v'}(t')}{\omega})\vec{E}_1(\vec{r'},t') + \frac{1}{\omega}(\vec{v'}(t')\cdot(\vec{E}_1(\vec{r'},t'))\vec{k}]$$

Since $\vec{E}_1 = \vec{E}_1 e^{i(\vec{k}\cdot\vec{r'}-\omega t')} = \vec{E}_1 e^{i(\vec{k}\cdot\vec{r'}-\omega(t'-t))}e^{-i\omega t}$

$$\begin{aligned}
\vec{k'}\cdot\vec{r'} &= \vec{k}\cdot\vec{r} + \frac{k_x}{\Omega}[v_{x0}\sin\Omega(t'-t) - v_{y0}(\cos\Omega(t'-t)-1)] \\
&\quad + \frac{k_y}{\Omega}[v_{x0}(\cos\Omega(t'-t)) - v_{y0}\sin\Omega(t'-t)] + k_z v_z(t'-t) \\
&= \vec{k}\cdot\vec{r} + \frac{1}{\Omega}(k_x v_{x0} + k_y v_{y0})\sin\Omega(t'-t) \\
&\quad - \frac{1}{\Omega}(k_x v_{y0} - k_y v_{x0})\cos\Omega(t'-t) \\
&\quad + \frac{1}{\Omega}(k_x v_{y0} - k_y v_{x0}) + k_z v_z(t'-t) \\
&= \vec{k}\cdot\vec{r} + \frac{k_\perp v_\perp}{\Omega}\sin(\Omega(t'-t)-\alpha) + \frac{k_\perp v_\perp}{\Omega}\sin\alpha + k_z v_z(t'-t)
\end{aligned}$$

$$\begin{aligned}
\vec{E}_1 e^{i(\vec{k}\cdot\vec{r'}-\omega t')} &= \vec{E}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\sum_n\sum_m J_m(\frac{k_\perp v_\perp}{\Omega})J_n(\frac{k_\perp v_\perp}{\Omega}) \times \exp[in(\Omega(t'-t)-\alpha)] \\
&\quad \times \exp[im\alpha]\exp[i(k_z v_z - \omega)(t'-t)]
\end{aligned}$$

where, we used $e^{ia\sin x} = \sum J_m(a)e^{imx}$

$$\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)+\frac{1}{\omega}\left(\vec{v'}\cdot\vec{E}_1\right)\vec{k}\right]\cdot\vec{\nabla}_{v'}f_0$$

$$\begin{aligned}
&= \frac{\partial f_0}{\partial v'_z}\left[\left(1-\frac{k_x v'_x}{\omega}-\frac{k_y v'_y}{\omega}-\frac{k_z v'_z}{\omega}\right)E_z+\left(v'_x E_x+v'_y E_y+v'_z E_z\right)\frac{k_z}{\omega}\right] \\
&\quad +\frac{\partial f_0}{\partial v'_\perp}\left[\left(1-\frac{k_x v'_x}{\omega}-\frac{k_y v'_y}{\omega}-\frac{k_z v'_z}{\omega}\right)\left(E_x\frac{v'_x}{v'_\perp}+E_y\frac{v'_y}{v'_\perp}\right)\right. \\
&\qquad \left.+\left(v'_x E_x+v'_y E_y+v'_z E_z\right)\left(\frac{k_x}{\omega}\frac{v'_x}{v'_\perp}+\frac{k_y}{\omega}\frac{v'_y}{v'_\perp}\right)v'_z\right]E_z \\
&= \left[\frac{\partial f_0}{\partial v_z}\frac{k_z v_\perp}{\omega}+\frac{\partial f_0}{\partial v_\perp}\left(1-\frac{k_z v_z}{\omega}\right)\right]\left[E_x\frac{v'_x}{v_\perp}+E_y\frac{v'_y}{v_\perp}\right] \\
&\qquad +\left[\frac{\partial f_0}{\partial v_z}\left(1-\frac{k_x v'_x}{\omega}-\frac{k_y v_y}{\omega}\right)+\frac{\partial f_0}{\partial v_\perp}\left(\frac{k_x}{\omega}\frac{v'_x}{v_\perp}+\frac{k_y}{\omega}\frac{v'_y}{v_\perp}\right)v'_z\right]E_z
\end{aligned}$$

$$\frac{v'_x}{v_\perp}=\cos\left(\alpha-\Omega\left(t'-t\right)\right),\qquad \frac{v'_y}{v_\perp}=\sin\left(\alpha-\Omega\left(t'-t\right)\right)$$

$$\begin{aligned}
&= \left[\frac{\partial f_0}{\partial v_z}\frac{k_z v_\perp}{\omega}+\frac{\partial f_0}{\partial v_\perp}\left(1-\frac{k_z v_z}{\omega}\right)\right]\left[E_x\cos\left(\alpha-\Omega(t'-t)\right)+E_y\sin\left(\alpha-\Omega(t'-t)\right)\right] \\
&\qquad +\left[\frac{\partial f_0}{\partial v_z}\left(1-\frac{k_\perp v_\perp}{\omega}\cos\left(\alpha-\Omega(t'-t)\right)\right)+\frac{\partial f_0}{\partial v_\perp}\frac{k_\perp v_z}{\omega}\cos\left(\alpha-\Omega(t'-t)\right)\right]E_z \\
&= C
\end{aligned}$$

$$\therefore f_1=\frac{q}{m}\sum_n\sum_m\int_{-\infty}^{t}J_m\left(\frac{k_\perp v_\perp}{\Omega}\right)J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{-i(n-m)\alpha}e^{-in\Omega(t'-t)}e^{i(k_z v_z-\omega)(t'-t)} \times C\,dt'$$

We dropped time dependence of $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$.

Let $t'-t=\tau$, $dt'=d\tau$, $\int_{-\infty}^{t}dt'=\int_{-\infty}^{0}d\tau$.

Thus,

$$f_1=-\frac{q}{m}\sum_n\sum_m e^{-i(n-m)\alpha}\int_{-\infty}^{0}d\tau\,J_m\left(\frac{k_\perp v_\perp}{\Omega}\right)J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{in\Omega\tau}e^{i(k_z v_z-\omega)\tau}\times C$$

A.1) For the E_x component

$$\cos(\alpha-\Omega\tau)=\frac{1}{2}\left(e^{i(\alpha-\Omega\tau)}+e^{-i(\alpha-\Omega\tau)}\right)$$

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{0}d\tau\,e^{in\Omega\tau}e^{i(k_z v_z-\omega)\tau}\left(e^{i\alpha}e^{-i\Omega\tau}+e^{-i\alpha}e^{i\Omega\tau}\right) \\
&= \frac{1}{2}\left[\frac{ie^{i\alpha}}{\omega-k_z v_z-(n-1)\Omega}+\frac{ie^{-i\alpha}}{\omega-k_z v_z-(n+1)\Omega}\right]
\end{aligned}$$

$$\begin{aligned}\therefore f_1 &= -\frac{iq}{m\omega}\sum_n\sum_m \frac{1}{2} J_m\left(\frac{k_\perp v_\perp}{\Omega}\right) J_n\left(\frac{k_\perp v_\perp}{\Omega}\right) \\ &\times\left[\frac{e^{-i[(n-1)-m]\alpha}}{\omega - k_z v_z - (n-1)\Omega} + \frac{e^{-i[(n+1)-m]\alpha}}{\omega - k_z v_z - (n+1)\Omega}\right] UE_x\end{aligned}$$

$$\begin{aligned} n-1 &\to n \quad \Rightarrow \quad J_n \to J_{n+1} \\ n+1 &\to n \quad \Rightarrow \quad J_n \to J_{n-1}\end{aligned}$$

$$\begin{aligned}&\sum_n \frac{1}{2} J_n\left(\frac{k_\perp v_\perp}{\Omega}\right)\left[\frac{e^{-i[(n-1)-m]\alpha}}{\omega - k_z v_z - (n-1)\Omega} + \frac{e^{-i[(n+1)-m]\alpha}}{\omega - k_z v_z - (n+1)\Omega}\right] \\ =& \sum_n \frac{e^{-i(n-m)\alpha}}{\omega - k_z v_z - n\Omega}\left(\frac{J_{n+1}+J_{n-1}}{2}\right) \\ =& \sum_n \frac{e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega}\,\frac{n}{\lambda}\, J_n(\lambda)\end{aligned}$$

where we used $\frac{nJ_n}{\lambda} = \frac{1}{2}(J_{n+1} + J_{n-1})$.

Thus,

$$f_1 = -\frac{iq}{m\omega}\sum_n\sum_m \frac{e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega}\,\frac{n}{\lambda}\, J_m(\lambda)\, J_n(\lambda)\, U\, E_x$$

where $\lambda = \frac{k_\perp v_\perp}{\Omega}$, $U = \frac{\partial f_0}{\partial v_z} k_z v_\perp + \frac{\partial f_0}{\partial v_\perp}(\omega - k_z v_z)$.

A.2) For the E_y component

$$\sin(\alpha - \Omega\tau) = \frac{1}{2i}\left(e^{i(\alpha-\Omega\tau)} - e^{-i(\alpha-\Omega\tau)}\right)$$

$$\begin{aligned}&\frac{1}{2i}\int_{-\infty}^{0} d\tau\, e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau}\left(e^{i(\alpha-\Omega\tau)} - e^{-i(\alpha-\Omega\tau)}\right) \\ =& \frac{1}{2i}\left[\frac{ie^{i\alpha}}{\omega - k_z v_z - (n-1)\Omega} - \frac{ie^{-i\alpha}}{\omega - k_z v_z - (n+1)\Omega}\right]\end{aligned}$$

similarly,

$$\begin{aligned} f_1 &= -\frac{iq}{m\omega}\sum_n\sum_m \frac{e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega}\, J_m(\lambda)\left(\frac{1}{i}\right)\left(-J_n'(\lambda)\right)\, U\, E_y \\ &= -\frac{iq}{m\omega}\sum_n\sum_m \frac{J_m(\lambda) e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega}\left(iJ_n'(\lambda)\right)\, U\, Ey\end{aligned}$$

where we used $2J_n' = J_{n-1} - J_{n+1}$.

A.3) For the E_z component

$$\sum_n \sum_m J_m J_n e^{-i(n-m)\alpha} \int_{-\infty}^{0} d\tau\, e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau} \left(1 - \frac{k_\perp v_\perp}{\omega} \cos(\alpha - \Omega\tau)\right)$$

$$= \sum_n \sum_n \frac{iJ_m J_n e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega} - \frac{k_\perp v_\perp}{\omega} \frac{iJ_m e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega} \frac{nJ_n}{\lambda}$$

$$= \sum_n \sum_m \frac{1}{\omega} \frac{iJ_m J_n e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega} (\omega - n\Omega)$$

$$\sum_n \sum_m J_m J_n e^{-i(n-m)\alpha} \int_{-\infty}^{0} d\tau e^{in\tau} e^{i(k_z v_z - \omega)\tau} \frac{k_z v_z}{\omega} \cos(\alpha - \Omega\tau)$$

$$= \sum_n \sum_m \left(\frac{1}{\omega}\right) \left(\frac{iJ_m J_n e^{-i(n-m)\alpha}}{\omega - k_z v_z - n\Omega}\right) \frac{n}{\lambda} i k_\perp v_z$$

$$= \sum_n \sum_m \frac{1}{\omega} \frac{iJ_m J_n e^{-i(n-m)\alpha}}{\omega - k_z v_z - n\Omega} \frac{n\Omega}{v_\perp} v_z$$

Thus,

$$f_1 = -\frac{iq}{m\omega} \sum_n \sum_m \frac{iJ_m(\lambda) e^{-i(n-m)\alpha}}{\omega - k_z v_z - n\Omega} \left[(\omega - n\Omega)\frac{\partial f_0}{\partial v_z} + \frac{n\Omega}{v_\perp} v_z \frac{\partial f_0}{\partial v_\perp}\right] \times E_z J_n(\lambda)$$

Putting together as components gives the perturbed distribution function f_1

$$\therefore f_1 = \frac{iq}{m\omega} \sum_n \sum_m \frac{iJ_m(\lambda) e^{-i(n-m)\alpha}}{\omega - k_z v_z - n\Omega} \left[-E_x \frac{n}{\lambda} U J_n - iE_y U J_n' - E_z W J_n\right]$$

where

$$\lambda = \frac{k_\perp v_\perp}{\Omega}, \quad J_n = J_n(\lambda), \quad J_n' = \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda)$$

$$U = (\omega - k_z v_z)\frac{\partial f_0}{\partial v_\perp} + k_z v_\perp \frac{\partial f_0}{\partial v_z}$$

$$W = \frac{n\Omega}{v_\perp} v_z \frac{\partial f_0}{\partial v_\perp} + (\omega - n\Omega)\frac{\partial f_0}{\partial v_z}$$

B. Calculation of $\int \underline{v} f_1 \mathrm{d}^3 v$

Let $\alpha = \phi$

B.1) $v_\perp \cos\phi \hat{x}$

$$\begin{aligned}
& \int \mathrm{d}v v_\perp \cos\phi f_1 \hat{x} = \int_\infty^\infty \mathrm{d}v_z \int_0^\infty v_\perp \mathrm{d}v_\perp \int_0^{2\pi} \mathrm{d}\phi v_\perp \cos\phi f_1 \hat{x} \\
= & \frac{iq}{m\omega} \sum_n \sum_n \int_\infty^\infty \mathrm{d}v_z \int_0^\infty v_\perp^2 \mathrm{d}v_\perp \int_0^{2\pi} \cos\phi e^{i(m-n)\phi} \mathrm{d}\phi \\
& \times \left[-E_x \frac{n}{\lambda} U J_n - i E_y U J_n' - E_z W J_n \right] \frac{J_m(\lambda)}{\omega - k_z v_z - n\Omega} \hat{x} \\
= & \frac{iq}{m\omega} \sum_n \sum_n \int_\infty^\infty \mathrm{d}v_z \int_0^\infty v_\perp^2 \mathrm{d}v_\perp \cdot 2\pi \left\{ \frac{\delta_{m,n+1} + \delta_{m,n-1}}{2} \right\} \\
& \times \left[-E_x \frac{n}{\lambda} U J_n - i E_y U J_n' - E_z W J_n \right] \frac{J_m(\lambda)}{\omega - k_z v_z - n\Omega} \hat{x} \\
= & \frac{iq}{m\omega} \sum_n \int_\infty^\infty \mathrm{d}v_z \int_0^\infty 2\pi v_\perp^2 \mathrm{d}v_\perp \left[-E_x \frac{n}{\lambda} U J_n - i E_y U J_n' - E_z W J_n \right] \\
& \times \frac{1}{\omega - k_z v_z - n\Omega} \left(\frac{J_{n+1} + J_{n-1}}{2} \right) \hat{x} \\
= & \frac{iq}{m\omega} \sum_n \int_\infty^\infty \mathrm{d}v_z \int_0^\infty 2\pi v_\perp \mathrm{d}v_\perp \frac{1}{\omega - k_z v_z - n\Omega} v_\perp \left(\frac{n}{\lambda} J_n \right) \\
& \times \left[-E_x \frac{n}{\lambda} U J_n - i E_y U J_n' - E_z W J_n \right] \hat{x} \\
= & \frac{-iq}{m\omega} \sum_n \int \mathrm{d}^3 v \frac{1}{\omega - k_z v_z - n\Omega} \\
& \times \left[v_\perp \left(\frac{n J_n}{\lambda} \right)^2 U E_x + i v_\perp \frac{n}{\lambda} J_n J_n' U E_y + v_\perp \frac{n}{\lambda} J_n^2 E_y \right] \hat{x}
\end{aligned}$$

B.2) $v_\perp \sin\phi \hat{y}$

$$\int_0^{2\pi} \sin\phi e^{i(m-n)\phi} \mathrm{d}\phi = 2\pi \frac{1}{2i} (-\delta_{m,n+1} + \delta_{m,n-1})$$

$$\begin{aligned}
& \int \mathrm{d}^3 v v_\perp \sin\phi f_1 \hat{y} \\
= & \frac{iq}{m\omega} \sum_n \int \mathrm{d}^3 v \frac{1}{\omega - k_z v_z - n\Omega} v_\perp \frac{1}{i} J_n' \\
& \times \left[-E_x \frac{n}{\lambda} U J_m - i E_y U J_n' - E_z W J_n \right] \hat{y} \\
= & -\frac{iq}{m\omega} \sum_n \int \mathrm{d}^3 v \frac{1}{\omega - k_z v_z - n\Omega} \\
& \times \left[-i v_\perp U \frac{n}{\lambda} J_n J_n' E_x + v_p erp U (J_n')^2 E_y + i v_\perp W J_n J_n' E_z \right] \hat{y}
\end{aligned}$$

B.3) $v_z \hat{z}$

$$\int_0^{2\pi} e^{i(m-n)\phi} \mathrm{d}\phi = 2\pi \delta_{n,m}$$

$$
\begin{aligned}
\therefore \quad & \int \mathrm{d}^3 v v_z f_1 \hat{z} \\
= \quad & \frac{iq}{m\omega} \sum_n \int \mathrm{d}^3 v \frac{1}{\omega - k_z v_z - n\Omega} v_z \cdot J_n \left[-E_x \frac{n}{\lambda} U J_n - i E_y U J_n' - E_z W J_n \right] \hat{z} \\
= \quad & \frac{iq}{m\omega} \sum_n \int \mathrm{d}^3 v \frac{1}{\omega - k_z v_z - n\Omega} \left[v_z \frac{n}{\lambda} J_n^2 E_x + i v_z J_n J_n' U E_y + v_z W J_n^2 E_z \right] \hat{z}
\end{aligned}
$$

Thus, the dielectric tensor, $\overleftrightarrow{K}$

$$
\overleftrightarrow{K} \cdot \vec{E} = \overleftrightarrow{1} \cdot \vec{E} + \frac{i}{\epsilon_0 \omega} \sum_s q_s \int \vec{v} f_{s1} d^3 v
$$

$$
\Longrightarrow \overleftrightarrow{1} + \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{1}{n_s} \sum_{n=-\infty}^{\infty} \int d^3 v \frac{\overleftrightarrow{S}}{\omega - k_z v_z - n\Omega} = \overleftrightarrow{K}
$$

where $\int d^3 v = \int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp$

$$
\overleftrightarrow{S} = \begin{pmatrix} v_\perp (\frac{nJ_n}{\lambda})^2 U & i v_\perp \frac{n}{\lambda} J_n J_n' U & v_\perp W \frac{n}{\lambda} J_n^2 \\ -i v_\perp U \frac{n}{\lambda} J_n J_n' & v_\perp U (J_n')^2 & -i v_\perp W J_n J_n' \\ v_z \frac{n}{\lambda} I J_n^2 & i v_z J_n J_n' U & v_z W J_n^2 \end{pmatrix}
$$

and,

$$
\omega_{ps}^2 = \frac{n_s q_s^2}{m_s \epsilon_0}
$$

$$
U = (\omega - k_z v_z) \frac{\partial f_{s0}}{\partial v_\perp} + k_z v_\perp \frac{\partial f_{s0}}{\partial v_z}
$$

$$
W = \frac{n\Omega_s}{v_\perp} v_z \frac{\partial f_{s0}}{\partial v_\perp} + (\omega - n\Omega_s) \frac{\partial f_{s0}}{\partial v_z}
$$

$$
\Omega_s = \frac{q_s B_0}{m_s}
$$

$$
\lambda_s = \frac{k_\perp v_\perp}{\Omega_s}
$$

C. For an isotropic Maxwellian plasma

$$f_{s0} = n_s \hat{f}_{s0} = n_s (a\sqrt{\pi})^{-3} e^{-v^2/a^2}$$

where the thermal velocity $a = \sqrt{2T_s/m_s}$. and n_s is the number density of the species s.
** Fried-Conte function

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-z} dx$$

and

$$Z'(z) = -2[1 + zZ(z)]$$

$$(1) \int_0^\infty e^{-a^2x^2} x J_n^2(px)dx = \frac{1}{2a^2} e^{-\frac{p^2}{2a^2}} I_n(\frac{p^2}{2a^2})$$

$$(2) \int_0^\infty e^{-a^2x^2} x^2 J_n'(px) J_n(px)dx = \frac{p}{4a^4} e^{-\frac{p^2}{2a^2}} [I_n'(\frac{p^2}{2a^2}) - I_n(\frac{p^2}{2n^2})]$$

$$(3) \int_0^\infty e^{-\frac{x^2}{2b}} x^3 [J_n'(x)]^2 dx = be^{-b}[n^2 I_n(b) - 2b^2(I_n'(b) - I_n(b))]$$

where I_n is the modified Bessel function of order n ;
I_n' denotes the derivative of I_n with respect to its argument.

$$\left(\star \int_0^\infty t J_\nu(pt) J_\nu(qt) e^{-a^2t^2} dt = \frac{1}{2a^2} exp(-\frac{p^2+q^2}{4a^2}) I_\nu(\frac{pq}{2a^2})\right)$$

when p=q,

$$(1) \int_0^\infty t J_n^2(pt) e^{-a^2t^2} dt = \frac{1}{2a^2} e^{-\frac{p^2}{2a^2}} I_n(\frac{p^2}{2a^2})$$

$$\frac{d}{dp}(1) \Rightarrow \int_0^\infty 2t^2 J_n(pt) J_n'(pt) e^{-a^2t^2} dt = \frac{1}{2a^2}(-\frac{p}{a^2}) e^{-\frac{p^2}{2a^2}} I_n(\frac{p^2}{2a^2}) + \frac{1}{2a^2} e^{-\frac{p^2}{2a^2}} (\frac{p}{a^2}) I_n'(\frac{p^2}{2a^2})$$

$$\Rightarrow (2) \int_0^\infty t^2 J_n(pt) J_n'(pt) e^{-a^2t^2} dt = \frac{p}{4a^4} e^{-\frac{p^2}{2a^2}} [I_n'(\frac{p^2}{2a^2}) - I_n(\frac{p^2}{2a^2})]$$

$$(3) \int_0^\infty e^{-\frac{x^2}{2b}} x^3 [J_n'(x)]^2 dx = be^{-b}[n^2 I_n(b) - 2b^2(I_n'(b) - I_n(b))]$$

Let $p = \frac{k_\perp}{\Omega} = \frac{\lambda}{v_\perp}$ and $a^2 \rightarrow \frac{1}{a^2} = \frac{m}{2T}, x = v_\perp, b = \frac{k_\perp^2 T}{m\Omega^2}$

$$(1)\frac{m}{2\pi T}\int_0^\infty 2\pi v_\perp J_n^2(\frac{k_\perp v_\perp}{\Omega})e^{-\frac{mv_\perp^2}{2T}} = e^{-b}I_n(b)$$

where $b = \frac{k_\perp^2 T}{m\Omega^2}$

$$\begin{aligned}(2)\frac{m}{2\pi T}\int_0^\infty 2\pi v_\perp^2 J_n(\frac{k_\perp v_\perp}{\Omega})J_n'(\frac{k_\perp v_\perp}{\Omega})e^{-\frac{mv_\perp^2}{2T}} &= \frac{m}{2\pi T}\int_0^\infty 2\pi v_\perp^2 J_n(\lambda)J_n'(\lambda)e^{-\frac{mv_\perp^2}{2T}} \\ &= \frac{k_\perp T}{m\Omega}e^{-b}[I_n'(b) - I_n(b)]\end{aligned}$$

$$(3)\frac{m}{2\pi T}\int_0^\infty 2\pi v_\perp^3 [J_n'(\lambda)]^2 e^{-\frac{mv_\perp^2}{2T}} = \frac{1}{2}\left(\frac{2T}{m}\right)e^{-b}\left[\frac{n^2}{b}I_n(b) + 2bI_n(b) - 2bI_n'(b)\right]$$

(* Note $\sum_n J_n^2 = 1; \sum_n e^{-b}I_n(b) = 1 \rightarrow \sum_n I_n(b) = e^b$)

$$\begin{aligned}U &= (\omega - k_z v_z)\frac{\partial f_{s0}}{\partial v_\perp} + k_z v_\perp \frac{\partial f_{s0}}{\partial v_z} \\ f_{s0} &= n_s(a\sqrt{\pi})^{-3}e^{-\frac{v_\perp^2 + v_z^2}{a^2}} \\ \rightarrow &\quad \frac{\partial f_{s0}}{\partial v_\perp} = -\frac{m_s n_s}{2\pi T}\frac{2}{a^3\sqrt{\pi}}v_\perp e^{-\frac{v_\perp^2 + v_z^2}{a^2}} = v_\perp A \\ \rightarrow &\quad \frac{\partial f_{s0}}{\partial v_z} = -\frac{m_s n_s}{2\pi T}\frac{2}{a^3\sqrt{\pi}}v_z e^{-\frac{v_\perp^2 + v_z^2}{a^2}} = v_z A \\ U &= (\omega - k_z v_z)\frac{\partial f_{s0}}{\partial v_\perp} + k_z v_\perp \frac{\partial f_{s0}}{\partial v_z} \\ &= \omega\frac{\partial f_{s0}}{\partial v_\perp} - k_z v_z v_\perp A + k_z v_z v_\perp A \\ &= \omega\frac{\partial f_{s0}}{\partial v_\perp} = \omega v_\perp A \\ W &= \frac{n\Omega}{v_\perp}v_\perp v_z A + (\omega - n\Omega)v_z A \\ &= n\Omega v_z A + (\omega - n\Omega)v_z A \\ &= \omega v_z A\end{aligned}$$

where

$$A = -\frac{m_s n_s}{2\pi T}\frac{2}{a^3\sqrt{\pi}}v_z e^{-\frac{v_\perp^2 + v_z^2}{a^2}}$$

D. Integration over velocity space

D.1 S_{xx} component

$$\int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp \frac{v_\perp \frac{n^2}{\lambda^2} J_n^2}{\omega - k_z v_z - n\Omega} \omega v_\perp A$$

- Integration over $v_\perp$

$$\begin{aligned}
& \int_0^{\infty} 2\pi v_\perp dv_\perp v_\perp \frac{n^2}{\lambda^2} J_n^2 \omega v_\perp A \\
= \; & n_s \frac{n^2\Omega^2}{k_\perp^2} \omega (-\frac{2}{a^3}) e^{-\frac{v_z^2}{a^2}} \frac{m}{2\pi T} \int_0^{\infty} 2\pi dv_\perp v_\perp J_n^2 e^{-\frac{v_\perp^2}{a^2}} \\
= \; & \frac{n^2\Omega^2}{k_\perp^2} \omega (-\frac{2}{a^3}) e^{-\frac{v_z^2}{a^2}} e^{-b} I_n(b)
\end{aligned}$$

- Integration over v_z

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega} &= -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{k_z v_z - (\omega - n\Omega)} \\
&= -\frac{1}{\sqrt{\pi}} \frac{1}{k_z} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - \frac{\omega - n\Omega}{k_z a}} \\
&= -\frac{1}{k_z} Z_n(\xi_n)
\end{aligned}$$

where $\xi_n = \frac{\omega - n\Omega}{k_z a}$ and $x = \frac{v_z}{a}$

$$\begin{aligned}
\therefore \int d^3v \frac{S_{xx}}{\omega - k_z v_z - n\Omega} &= n_s \frac{n^2\Omega^2}{k_\perp^2} \frac{2\omega}{a^3 k_z} e^{-b} I_n(b) Z_n(\xi_n) \\
&= n_s \frac{\omega}{a k_z} \frac{n^2\Omega^2}{k_\perp^2} \frac{2}{a^2} e^{-b} I_n(b) Z_n(\xi_n) \\
&= n_s \frac{\omega}{a k_z} \frac{n^2\Omega^2 m}{k_\perp^2 T} e^{-b} I_n(b) Z_n(\xi_n) \\
&= n_s \frac{\omega}{a k_z} \frac{n^2}{b} e^{-b} I_n(b) Z_n(\xi_n)
\end{aligned}$$

Thus

$$K_{xx} = 1 + \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n \frac{n^2}{b} e^{-b} I_n(b) Z_n(\xi_n)$$

D.2 S_{xy} component

$$\int d^3v \frac{i v_\perp \frac{n}{\lambda} J_n J_n^{'} U}{\omega - k_z v_z - n\Omega}$$

- Integration over $v_\perp$

$$\begin{aligned}
\int_0^\infty 2\pi v_\perp dv_\perp v_\perp \frac{n}{\lambda} J_n J_n' \omega v_\perp A &= n_s \frac{n\Omega\omega}{k_\perp} \int_0^\infty 2\pi v_\perp^2 dv_\perp J_n J_n' \omega A \\
&= n_s \frac{n\Omega\omega}{k_\perp} \frac{m}{2\pi T} (-\frac{2}{a^3}) e^{-\frac{v_z^2}{a^2}} \int_0^\infty 2\pi v_\perp^2 dv_\perp J_n J_n' \omega e^{-\frac{v_\perp^2}{a^2}} \\
&= n_s \frac{n\Omega\omega}{k_\perp} (-\frac{2}{a^3}) e^{-\frac{v_z^2}{a^2}} \frac{k_\perp T}{m\Omega} e^{-b} [I_n'(b) - I_n(b)] \\
&= n_s \frac{2T}{m} (\frac{n\omega}{a^3}) e^{-b} [I_n'(b) - I_n(b)] e^{-\frac{v_z^2}{a^2}} \\
&= n_s \frac{-n\omega}{a} e^{-b} [I_n'(b) - I_n(b)] e^{-\frac{v_z^2}{a^2}}
\end{aligned}$$

- Integration over v_z

$$\begin{aligned}
\tfrac{1}{\sqrt{\pi}} & \int_{-\infty}^\infty dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega} = -\frac{Z_n(\xi)}{k_z} \\
\therefore & \int d^3 v \frac{S_{xy}}{\omega - k_z v_z - n\Omega} \\
= \ & i n_s \frac{\omega}{k_z a} n e^{-b} [I_n'(b) - I_n(b)] Z_n(\xi_n)
\end{aligned}$$

Thus

$$K_{xy} = -K_{yx} = i \frac{\omega_{ps}^2}{\omega^2} \frac{-\omega}{k_z a} \sum_n n e^{-b} [I_n'(b) - I_n(b)] Z_n(\xi_n)$$

D.3 S_{xz} component

$$\int d^3 v \frac{v_\perp W \frac{n}{\lambda} J_n^2}{\omega - k_z v_z - n\Omega} = \int d^3 v \frac{v_\perp \omega v_z \frac{n}{\lambda} J_n^2 A}{\omega - k_z v_z - n\Omega}$$

$$= \int dv_z \frac{\omega v_z}{\omega - k_z v_z - n\Omega} \int 2\pi v_\perp dv_\perp v_\perp \frac{n}{\lambda} J_n^2 A$$

- Integration over $v_\perp$

$$\frac{2}{a^3} e^{-\frac{v_z^2}{a^2}} \frac{n\Omega}{k_\perp} \left(-\frac{m_s n_s}{2\pi T}\right) \int_0^\infty 2\pi v_\perp dv_\perp J_n^2 e^{-\frac{v_\perp^2}{a^2}}$$

$$= -\frac{2n_s}{a^3} e^{-\frac{v_z^2}{a^2}} \frac{n\Omega}{k_\perp} e^{-b} I_n(b)$$

- Integration over v_z

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dv_z \frac{v_z}{\omega - k_z v_z - n\Omega} e^{-\frac{v_z^2}{a^2}} &= -\frac{1}{k_z}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{v_z e^{-\frac{v_z^2}{a^2}}}{v_z - \frac{\omega-n\Omega}{k_z}} dv_z \\
&= -\frac{1}{k_z}\frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{xe^{-x^2}}{x-\zeta_n}dx \\
&= -\frac{a}{k_z}\left[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}dx + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\zeta_n e^{-x^2}}{x-\zeta_n}dx\right] \\
&= -\frac{a}{k_z}\left[1+\zeta_n Z_n(\zeta_n)\right] = \frac{a}{2k_z}Z_n'
\end{aligned}$$

$$\begin{aligned}
\therefore \int d^3v \frac{v_\perp W \frac{n}{\lambda} J_n^2}{\omega - k_z v_z - n\Omega} &= -\frac{2\omega n_s}{a^3}\frac{n\Omega}{k_\perp}\frac{a}{2h_z} e^{-b} I_n Z_n' \\
&= -\frac{\omega n_s}{k_z a}\frac{n\Omega}{k_\perp}\frac{1}{a} e^{-b} I_n Z_n' \\
&= -\frac{\omega n_s}{k_z a}\frac{n\Omega}{k_\perp}\sqrt{\frac{m}{2T}} e^{-b} I_n Z_n' \\
&= -\frac{\omega n_s}{k_z a}\left(\pm\frac{n}{\sqrt{2b}}\right) e^{-b} I_n Z_n' \qquad \begin{pmatrix} +\text{sign} & : & \text{ion} \\ -\text{sign} & : & \text{electron}\end{pmatrix}
\end{aligned}$$

Thus

$$K_{xz} = -\frac{\omega_p^2}{\omega^2}\frac{\omega}{k_z a}\sum_n \left(\pm\frac{n}{\sqrt{2b}}\right) e^{-b} I_n Z_n'$$

since

$$S_{zx} = v_z \frac{n}{\lambda} U J_n^2 = \frac{n}{\lambda}\omega v_\perp v_z J_n^2 A = S_{xz}$$

$$K_{zx} = K_{xz} = -\frac{\omega_p^2}{\omega^2}\frac{\omega}{k_z a}\sum_n \left(\pm\frac{n}{\sqrt{2b}}\right) e^{-b} I_n Z_n'$$

D.4 S_{yy} component

$$\int d^3v \frac{v_\perp U [J_n']^2}{\omega - k_z v_z - n\Omega} = \int d^3v \frac{\omega v_\perp^2 [J_n']^2 A}{\omega - k_z v_z - n\Omega}$$

- Integration over $v_\perp$

$$\frac{2n_s}{a^3}\left(-\frac{m}{2\pi T}\right)\int 2\pi v_\perp dv_\perp v_\perp^2 (J_n')^2 e^{-\frac{v_\perp^2}{a^2}} = \left(-\frac{2n_s}{a^3}\right)\frac{1}{2}a^2 e^{-b}\left[\frac{n^2}{b}I_n(b) - 2b(I_n' - I_n)\right]$$

- Integration over v_z

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega} = -\frac{1}{k_z} Z_n(\zeta_n)$$

$$\therefore \int d^3v \frac{v_\perp U[J_n']^2}{\omega - k_z v_z - n\Omega} = \frac{1}{k_z}\frac{2\omega n_s}{a^3}\frac{1}{2}a^2 e^{-b}\left[\frac{n^2}{b}I_n - 2b(I_n' - I_n)\right] Z_n$$
$$= \frac{\omega n_s}{k_z a} e^{-b}\left[\frac{n^2}{b}I_n - 2b(I_n' - I_n)\right] Z_n$$

Thus

$$K_{yy} = 1 + \frac{\omega_{ps}^2}{\omega^2}\frac{\omega}{k_z a}\sum_n e^{-b}\left[\frac{n^2}{b}I_n - 2b(I_n' - I_n)\right] Z_n$$

D.5 S_{yz} component

$$S_{yz} = -iv_\perp W J_n J_n' = -i\omega v_\perp v_z A J_n J_n'$$

$$S_{zy} = +iv_z U J_n J_n' = +i\omega v_\perp v_z A J_n J_n' = -S_{yz}$$

$$\int d^3v \frac{i\omega v_\perp v_z A J_n J_n'}{\omega - k_z v_z - n\Omega}$$

- Integration over $v_\perp$

$$\frac{2}{a^3}\left(-\frac{m_s n_s}{2\pi T}\right)\int 2\pi v_\perp dv_\perp v_\perp J_n J_n' e^{-\frac{v_\perp^2}{a^2}} = \left(-\frac{2n_s}{a^3}\right)\frac{k_\perp T}{m\Omega}e^{-b}(I_n' - I_n)$$

- Integration over v_z

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dv_z \frac{v_z e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega} = \frac{a}{2k_z}Z_n' \Longleftarrow \text{ we already calculated.}$$

$$\therefore \quad \int d^3v \frac{-i\omega v_\perp v_z I_n I_n'}{\omega - k_z v_z - n\Omega} = i\frac{2\omega n_s}{a^2}\frac{k_\perp T}{m\Omega}e^{-b}(I_n' - I_n)\frac{a}{2k_z}Z_n'$$
$$= i\frac{\omega n_s}{k_z a}\sqrt{\frac{m}{2T}}\frac{k_\perp T}{m\Omega}e^{-b}(I_n' - I_n)Z_n'$$
$$= i\frac{\omega n_s}{k_z a}\left(\pm\sqrt{\frac{b}{2}}\right)\frac{k_\perp T}{m\Omega}e^{-b}(I_n' - I_n)Z_n'$$

Thus,

$$K_{yz} = -K_{zy} = i\frac{\omega_{ps}^2}{\omega^2}\frac{\omega}{k_z a}\sum_n\left(\pm\sqrt{\frac{b}{2}}\right)e^{-b}(I_n' - I_n)Z_m'$$

D.6 S_{zz} component

$$\int d^3v \frac{v_z W J_n^2}{\omega - k_z v_z - n\Omega} = \int d^3v \frac{\omega v_z^2 J_n^2 A}{\omega - k_z v_z - n\Omega}$$

- Integration over $v_\perp$

$$\left(-\frac{2n_s}{a^3}\right) \frac{m}{2\pi T} \int_0^\infty dv_\perp 2\pi J_n^2 e^{-\frac{v_\perp^2}{a^2}} = -\frac{2n_s}{a^3} e^{-b} I_n(b)$$

- Integration over v_z

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \frac{v_z^2 e - \frac{v_z^2}{a^2}}{\omega - k_z v_z - n\Omega} &= -\frac{1}{k_z\sqrt{\pi}} \int_{-\infty}^{\infty} dv_z \frac{v_z^2 e^{-\frac{v_z^2}{a^2}}}{v_z - \frac{(\omega - n\Omega)}{k_z}} \\ &= -\frac{a^2}{k_z\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{x^2 e^{-x^2}}{x - \zeta_n} \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} dx \frac{x^2 e^{-x^2}}{x - \zeta_n} &= \int_{\infty}^{\infty} dx \frac{x^2 - \zeta_n^2 + \zeta_n^2}{x - \zeta_n} e^{-x^2} dx \\ &= \int_{-\infty}^{\infty} (x + \zeta_n) e^{-x^2} dx + \zeta_n^2 \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta_n} dx \\ &= \zeta_n \sqrt{\pi} + \sqrt{\pi} \zeta_n^2 Z_n = \zeta_n \sqrt{\pi} (1 + \zeta_n Z_n) \end{aligned}$$

$$\begin{aligned} \therefore \qquad -\frac{a^2}{k_z\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{x^2 e - x^2}{x - \zeta_n} &= -\frac{a^2}{k_z} \zeta_n (1 + \zeta_n Z_n) \\ &= \frac{a^2}{2k_z} \zeta_n Z_n' \end{aligned}$$

$$\begin{aligned} \therefore \qquad \int d^3v \frac{v_z W I_n^2}{\omega - k_z v_z - n\Omega} &= -\frac{2\omega n_s}{a^3} \frac{a^2}{2k_z} \zeta_n e^{-b} I_n Z_n' \\ &= -\frac{\omega n_s}{k_z a} e^{-b} I_n \zeta_n Z_n' \end{aligned}$$

Thus,

$$K_{zz} = 1 - \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n e^{-b} I_n \zeta_n Z_n'$$

Therefore, the dielectric tensor,

$$\overset{\leftrightarrow}{K} = \begin{pmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{pmatrix}$$

where

$$\begin{aligned}
K_{xx} &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n \frac{n^2}{b} e^{-b} I_n Z_n \\
K_{yy} &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n \{ \frac{n^2}{b} e^{-b} I_n Z_n - 2be^{-b}(I_n' - I_n) Z_n \} \\
K_{zz} &= 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n e^{-b} I_n \zeta_n Z_n' \\
K_{xy} &= -K_{yx} = i \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n n e^{-b} (I_n' - I_n) Z_n \\
K_{xz} &= K_{zx} = -\sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n \left(\pm \frac{n}{\sqrt{2b}} \right) e^{-b} I_n Z_n' \\
K_{yz} &= -K_{zy} = i \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega}{k_z a} \sum_n \left(\pm \frac{b}{\sqrt{2}} \right) e^{-b} (I_n' - I_n) Z_n'
\end{aligned}$$

where,

$$\begin{aligned}
a &= \sqrt{\frac{2T}{m}} \\
b &= \frac{k_\perp^2 T}{m\Omega^2} \text{ is the argument of } I_n \\
\zeta_n &= \frac{\omega - n\Omega}{k_z a} \text{ is the argument of } Z_n
\end{aligned}$$

$\Omega = \frac{q_s B_0}{m} \Longrightarrow$ sign contained, Z' denotes the derivative of Z_n with respect to its argument, and the summation run for all integer n.

3.2 Electrostatic Dispersion Relation

$$K(\vec{k},\omega) \rightarrow \epsilon = 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \sum_n \int d^3v \frac{J_n^2}{\omega - k_\| v_\| - n\Omega_s}[\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + k_\| \frac{\partial f_{s0}}{\partial v_\|}]$$

[Harris Dispersion Relation]

- Derivation:

Maxwell's equations for Electrostatic:

$$\begin{aligned} \vec{E}_1 &= -\nabla\phi_1 = -i\vec{k}\phi_1 \\ \nabla\times\vec{E}_1 &\simeq 0 \Rightarrow B_1 = 0 \\ \nabla\cdot\vec{E}_1 &= \frac{1}{\epsilon_0}\sum_s q_s \int f_{s1} d^3v \Rightarrow i\vec{k}\cdot\vec{E}_1 = \frac{1}{\epsilon}\sum_s q_s \int f_{s1} d^3v \end{aligned}$$

First-order Vlasov equation :

$$\frac{\partial f_{s1}}{\partial t} + \vec{v}\cdot\nabla f_{s1} + (\frac{q_s}{m_s}\vec{v}\times\vec{B}_0)\cdot\nabla_v f_{s1} = -\frac{q_s}{m_s}\vec{E}_1\cdot\nabla_v f_{s0}$$

$$\Rightarrow \frac{df_{s1}}{dt} = S(\vec{r},v,t) = -\frac{q_s}{m_s}\vec{E}_1\cdot\nabla_v f_{s0}$$

$$\therefore f_{s1} = -\frac{q_s}{m_s}\int_{-\infty}^{t}\vec{E}_1(\vec{r'},t)\cdot\nabla_{v'} f_{s0} dt' \qquad (2.2.1)$$

$\vec{E}_1$ has fourier component of $e^{i(\vec{k}\cdot\vec{r'}-\omega t')}$
From section 3.1,

$$\vec{k}\cdot\vec{r'} = \vec{k}\cdot\vec{r} + \frac{k_\perp v_\perp}{\Omega}\sin(\Omega(t'-t)-\alpha) + \frac{k_\perp v_\perp}{\Omega}\sin\alpha + k_z v_z (t'-t)$$

Then,

$$\begin{aligned} \vec{E}_1 = \vec{E}_1(\vec{r})e^{i(\vec{k}\cdot\vec{r'}-\omega t')} &= \vec{E}_1(\vec{r})e^{i(\vec{k}\cdot\vec{r}-\omega t')}\sum_n\sum_m J_n(\frac{k\perp v_\perp}{\Omega})J_m(\frac{k\perp v_\perp}{\Omega}) \\ &\quad \times\exp[in(\Omega(t'-t)-\alpha)]\exp[im\alpha]\exp[i(k_z v_z-\omega)(t'-t)] \\ &= \vec{E}_1(\vec{r})T(\vec{r},\vec{v},t'). \end{aligned}$$

where $T(\vec{r},\vec{v},t') = e^{i(\vec{k}\cdot\vec{r}-\omega t)}\sum_n\sum_m J_n J_m e^{in(\Omega(t'-t)-\alpha)}e^{im\alpha}e^{i(k_z v_z-\omega)(t'-t)}$

and we used $e^{i\alpha\sin x} = \sum_{m=-\infty}^{\infty} J_m(a)e^{imx}$

The integrand of Eq.(2.2.1) is

$$\begin{aligned} \vec{E}_1(\vec{r'},t')\cdot\nabla_{v'} f_{s0} &= T\vec{E}_1(\vec{r})\cdot\nabla_{v'} f_{s0} \\ &= T[E_z\frac{\partial f_{s0}}{\partial v'_z} + (E_x\frac{v'_x}{v'_\perp} + E_y\frac{v'_y}{v'_\perp})\frac{\partial f_{s0}}{\partial v'_\perp}] \\ &= T\{E_z\frac{\partial f_{s0}}{\partial v'_z} + [E_x\cos(\alpha-\Omega(t'-t)) + E_y\sin(\alpha-\Omega(t'-t))]\frac{\partial f_{s0}}{\partial v'_\perp}\} \end{aligned}$$

We dropped sub index 1.
Note that

$$\begin{aligned} v'_x &= v'_\perp \cos(\alpha - \Omega(t'-t)) = v_\perp \cos(\alpha - \Omega(t'-t)) \\ v'_y &= v'_\perp \sin(\alpha - \Omega(t'-t)) = v_\perp \sin(\alpha - \Omega(t'-t)) \end{aligned}$$

Let $t' - t = \tau,\ dt' = d\tau,\ \int_{-\infty}^{t} dt' = \int_{-\infty}^{0} d\tau$

Thus Eq.(2.2.1) becomes

$$\begin{aligned} f_{s1} &= -\frac{q_s}{m_s}\sum_n\sum_m e^{i(m-n)\alpha} J_n(\frac{k_\perp v_\perp}{\Omega}) J_m(\frac{k_\perp v_\perp}{\Omega}) \\ &\times \int_{-\infty}^{0} d\tau e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau}\{E_z \frac{\partial f_{s0}}{\partial v_z} + [E_x \cos(\alpha - \Omega\tau) + E_y \sin(\alpha - \Omega\tau)]\frac{\partial f_{s0}}{\partial v_\perp}\} \end{aligned}$$

Integration for E_x component gives

$$\int_{-\infty}^{0} d\tau e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau} \cos(\alpha - \Omega\tau) = \frac{1}{2}[\frac{ie^{i\alpha}}{\omega - k_z v_z - (n-1)\Omega} + \frac{ie^{-i\alpha}}{\omega - k_z v_z - (n+1)\Omega}]$$

Integration for E_y component gives

$$\int_{-\infty}^{0} d\tau e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau} \sin(\alpha - \Omega\tau) = \frac{1}{2i}[\frac{ie^{i\alpha}}{\omega - k_z v_z - (n-1)\Omega} - \frac{ie^{-i\alpha}}{\omega - k_z v_z - (n+1)\Omega}]$$

Integration for E_z component gives

$$\int_{-\infty}^{0} d\tau e^{in\Omega\tau} e^{i(k_z v_z - \omega)\tau} = \frac{1}{i}\frac{-1}{\omega - k_z v_z - n\Omega} \qquad (2.2.2)$$

$$\sum_n \frac{1}{2} J_n\left(\frac{k_\perp v_\perp}{\Omega_s}\right)\left[\frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - (n-1)\Omega_s} + \frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - (n+1)\Omega_s}\right]$$

$$\begin{aligned} (n-1 &\to n \Rightarrow J_n \to J_{n+1}) \\ (n+1 &\to n \Rightarrow J_n \to J_{n-1}) \end{aligned}$$

$$\begin{aligned} &= \sum_n \frac{1}{2}\frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - n\Omega_s}(J_{n+1} + J_{n-1}) \\ &= \sum_n \frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - n\Omega_s}\frac{n\Omega}{k_\perp v_\perp} J_n(\frac{k_\perp v_\perp}{\Omega_s}) \qquad (2.2.3) \end{aligned}$$

Where, we used

$$\frac{J_n}{x} = \frac{1}{2}\left(J_{n+1}(x) + J_{n-1}(x)\right)$$

and

$$\sum_n \frac{1}{2i} J_n\left(\frac{k_\perp v_\perp}{\Omega_s}\right)\left[\frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - (n-1)\Omega_s} - \frac{e^{i(m-(n+1))\alpha}}{\omega - k_z v_z - (n+1)\Omega_s}\right]$$

$$= \sum_n \frac{1}{2i} \frac{e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega_s} (J_{n+1} + J_{n-1})$$

$$= \sum_n (\frac{1}{i}) \frac{e^{i(m-(n-1))\alpha}}{\omega - k_z v_z - n\Omega_s} (-J_n') \qquad (2.2.4)$$

Where, we used

$$2J_n' = J_{n-1} - J_{n+1}$$

Thus, putting the result of integration, Eqs. (2.2.2)-(2.2.4)

$$f_{s1} = -\frac{iq_s}{m_s} \sum_n \sum_m J_m(\frac{k_\perp v_\perp}{\Omega_s}) \frac{e^{i(m-n)\alpha}}{\omega - k_z v_z - n\Omega_s}$$

$$\times [E_x J_n \frac{n\Omega_s}{k_\perp v_\perp} \frac{\partial f_{s0}}{\partial v_\perp} \ + \ E_y J_n' \frac{\partial f_{s0}}{\partial v_\perp} \ + \ E_z J_n \frac{\partial f_{s0}}{\partial v_z}]$$

The electrostatic dispersion relation comes from

$$i\vec{k} \cdot \vec{E}_1 = \frac{1}{\epsilon_0} \sum_s q_s \int f_{s1} d^3v \quad \Rightarrow \ k^2\phi = \frac{1}{\epsilon_0} \sum_s q_s \int f_{s1} d^3v$$

$$\Rightarrow \ k^2\phi - \frac{1}{\epsilon} \sum_s q_s \int f_{s1} d^3v = \epsilon k^2 \phi = 0$$

$$\underline{\int f_{s1} d^3v = \ ?}$$

$$\int f_{s1} d^3v = \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_\perp dv_\perp \int_0^{2\pi} d\alpha f_{s1}$$

Since f_{s1} has the dependence of $e^{i(m-n)\alpha}$, the integration of $d\alpha$ gives $2\pi\delta_{mn}$.

$$\int f_{s1} d^3v = -\frac{iq_s}{m_s} \sum_n J_n \int_{-\infty}^{\infty} dv_z \frac{1}{\omega - k_z v_z - n\Omega_s} \int_0^{\infty} 2\pi v_\perp dv_\perp [E_x \frac{n\Omega_s}{k_\perp v_\perp} J_n \frac{\partial f_{s0}}{\partial v_\perp}$$
$$+ E_y J_n' \frac{\partial f_{s0}}{\partial v_\perp} + E_z J_n' \frac{\partial f_{s0}}{\partial v_z}]$$

But, $E_x = -ik_x\phi, \ E_y = -ik_y\phi, \ E_z = -ik_z\phi$ (2.2.5)

Assume $\vec{k} = k_x\vec{x} + k_z\vec{z} = k_\perp\vec{x} + k_z\vec{z} \ (k_y = 0)$ (2.2.6)

Now, we define the integration in velocity space as

$$\int d^3v = \int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp \tag{2.2.7}$$

Then, from Eqs.(2.2.5)-(2.2.7)

$$\int f_{s1} d^3v = -\frac{q_s\phi}{m_s}\sum_n \int d^3v \frac{J_n^2}{\omega - k_z v_z - n\Omega_s}[\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + k_z \frac{\partial f_{s0}}{\partial v_z}]$$

$$\epsilon k^2\phi = k^2\phi - \frac{1}{\epsilon_0}\sum_s q_s \int d^3v f_{s1}$$

$$= k^2\phi + \sum_s \frac{q_s\phi}{m_s\epsilon_0}\sum_n \int d^3v \frac{J_n^2}{\omega - k_z v_z - n\Omega_s}[\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + k_z \frac{\partial f_{s0}}{\partial v_z}]$$

$$= k^2\phi + \sum_s \omega_{ps}^2\phi\sum_n \int d^3v \frac{J_n^2}{\omega - k_z v_z - n\Omega_s}[\frac{n\Omega_s}{v_\perp}\frac{\partial \hat{f}_{s0}}{\partial v_\perp} + k_z \frac{\partial \hat{f}_{s0}}{\partial v_z}]$$

(Note : $\hat{f}_0$ is the normalized distribution function)

$$\therefore \epsilon = 1 + \sum_s \frac{\omega_{ps}^2}{k^2}\sum_n \int d^3v \frac{J_n^2\left(\frac{k_\perp v_\perp}{\Omega_s}\right)}{\omega - k_z vz - n\Omega_s}\left(\frac{n\Omega_s}{v_\perp}\frac{\partial \hat{f}_0}{\partial v_\perp} + k_z \frac{\partial \hat{f}_0}{\partial v_z}\right) \tag{2.2.8}$$

- <u>The E.S. Dispersion Relation for M-B distribution</u>

$$\hat{f}_0 = (a\sqrt{\pi})^{-3} e^{-\frac{v^2}{a^2}} \quad \left(a = \sqrt{\frac{2T_s}{m_s}}\right)$$

$$* \int_{\infty}^{\infty} e^{-s^2x^2} x J_n^2(px)dx = \frac{1}{2s^2} e^{-\frac{p^2}{2a^2}} I_n(\frac{p^2}{2s^2})$$

$$\left(\begin{array}{lll} \frac{\partial \hat{f}_0}{\partial v_\perp} & = & \frac{\partial}{\partial v_\perp}\left[(a\sqrt{\pi})^{-3} e^{-(\frac{v_\perp^2}{a^2}+\frac{v_z^2}{a^2})} \right] \\ & = & (a\sqrt{\pi})^{-3}\left(-\frac{2v_\perp}{a^2}\right) e^{-\frac{v^2}{a^2}} = -\frac{2}{a^5\pi\sqrt{\pi}} e^{-\frac{v^2}{a^2}} \\ \frac{\partial \hat{f}_0}{\partial v_z} & = & -\frac{2}{a^5\pi\sqrt{\pi}} v_z e^{-\frac{v^2}{a^2}} \end{array} \right.$$

$$
\begin{aligned}
&\int d^3v \frac{J_n^2(\frac{k_\perp v_\perp}{\Omega_s})}{\omega - k_z v_z - n\Omega_s}\left[\frac{n\Omega_s}{v_\perp}\left(-\frac{2}{a^5\pi\sqrt{\pi}}v_\perp e^{-v^2}a^2\right) + k_z\left(-\frac{2}{a^5\pi\sqrt{\pi}}v_z e^{-\frac{v^2}{a^2}}\right)\right] \\
= \;& -\frac{2}{a^5\pi\sqrt{\pi}}\int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp \frac{J_n^2(k_\perp v_\perp)}{\omega - k_z v_z - n\Omega_s}(n\Omega_s + k_z v_z)e^{-\frac{v^2}{a^2}} \\
= \;& \frac{4}{a^5\sqrt{\pi}}\int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega_s}(n\Omega_s + k_z v_z)\left[\frac{1}{2}a^2 e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp^2}{2\Omega_s^2}\right)\right] \\
= \;& -\frac{2}{a^3\sqrt{\pi}}e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp^2}{2\Omega_s^2}\right)\int_{-\infty}^{\infty} dv_z \frac{n\Omega_s + k_z v_z}{\omega - k_z v_z - n\Omega_s}e^{-\frac{v_z^2}{a^2}}
\end{aligned}
$$

The first term of the integrand:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{\omega - k_z v_z - n\Omega_s} &= n\Omega_s\left(-\frac{1}{k_z}\right)\int_{-\infty}^{\infty} dv_z \frac{e^{-\frac{v_z^2}{a^2}}}{v_z - \frac{\omega - n\Omega_s}{k_z}} \\
&= \frac{n\Omega_s}{k_z}\int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - \frac{\omega - n\Omega_s}{k_z a}} \\
&= -\frac{n\Omega_s}{k_z}\sqrt{\pi}Z_n(\zeta_n)
\end{aligned}
$$

where $Z_n(\zeta_n)$ is the dispersion function and its argument $\zeta_n = \frac{\omega - n\Omega_s}{k_z a}$.

The second term of the integrand:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dv_z \frac{k_z v_z}{\omega - k_z v_z - n\Omega_z}\, e^{-\frac{v_z^2}{a^2}} &= -\int_{-\infty}^{\infty} dv_z \frac{v_z}{v_z - \frac{\omega - n\Omega_s}{k_z}}e^{-\frac{v_z^2}{a^2}} \\
&= -a\int_{-\infty}^{\infty} dx \frac{xe^{-x^2}}{x - \zeta_n} \\
&= -a\int_{\infty}^{\infty} dx \frac{(x - \zeta_n + \zeta_n)}{x - \zeta_n}e^{-x^2} \\
&= -a\left[\int_{-\infty}^{\infty} dx e^{-x^2} + \zeta_n\int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - \zeta_n}\right] \\
&= -a\left(\sqrt{\pi} + \sqrt{\pi}\zeta_n Z_n(\zeta_n)\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\int d^3v \; \frac{J_n^2}{\omega - k_z v_z - n\Omega_s}\left[\frac{n\Omega_s}{v_\perp}\frac{\partial \hat{f}_0}{\partial v_\perp} + k_z\frac{\partial \hat{f}_0}{\partial v_z}\right] \\
= \;& -\frac{2}{a^3\sqrt{\pi}}\; e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp}{2\Omega_s^2}\right)\left[-\frac{n\Omega_s}{k_z}\sqrt{\pi}Z_n(\zeta_n) - a\sqrt{\pi} - a\sqrt{\pi}\zeta Z_n(\zeta_n)\right] \\
= \;& \frac{2}{a^2}\; e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp}{2\Omega_s^2}\right)\left[1 + \left(\frac{n\Omega_s}{k_z a} + \frac{\omega - n\Omega_s}{k_z a}\right)Z_n(\zeta_n)\right] \\
= \;& \frac{2}{a^2}\; e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp}{2\Omega_s^2}\right)\left[1 + \frac{\omega}{k_z a}Z_n(\zeta_n)\right]
\end{aligned}
$$

Thus, the dielectric constant ϵ is

$$\begin{aligned}\epsilon &= 1+\sum_s \frac{\omega_{ps}^2}{k^2}\frac{2}{a^2}\sum_n e^{-\frac{a^2k_\perp^2}{2\Omega_s^2}} I_n\left(\frac{a^2k_\perp}{2\Omega_s^2}\right)\left[1+\frac{\omega}{k_z a}Z_n(\zeta_n)\right] \\ &= 1+\sum_s \frac{1}{k^2\lambda_{D_s}^2}\sum_n e^{-b}I_n(b)\left[1+\frac{\omega}{k_z a}Z_n(\zeta_n)\right] \qquad (2.2.9)\end{aligned}$$

Where

$$\begin{aligned}\lambda_{D_s}^{-2} &= \frac{n_e q_s^2}{\epsilon_0 T_s} = \frac{2\omega_{ps}^2}{a^2} \quad \text{: Debye Length} \\ b &= \frac{a^2k_\perp^2}{2\Omega_s^2} = k_\perp^2\left(\frac{T_s}{m_s\Omega_s^2}\right) = k_\perp^2\rho^2 \\ \zeta_n &= \frac{\omega - n\Omega_s}{k_z a} \\ a &= \sqrt{\frac{2T_s}{m_s}}\end{aligned}$$

$\therefore$ The dispersion relation:

$$\epsilon = 0 = 1+\sum_s \frac{1}{k^2\lambda_{D_s}^2}\sum_n e^{-b}I_n(b)\left[1+\frac{\omega}{k_z a}Z_n(\zeta_n)\right] \qquad (2.2.10)$$

3.2.1 Electrostatic Modes in Hot Plasma

A. Electron Modes ($k_z \neq 0,\ \omega \gg \omega_{pi},\ \Omega_i$, low temperature)

$$\omega \gg \omega_{pi}, \Omega_i$$

$$\begin{aligned}\text{For } B_0 &= 3T \\ n_e &= 1.0\times 10^{14} cm^{-3} \\ \Omega_e &= 84GHz \\ \omega_{pe} &= 90GHz \\ \omega_{pi} &= 2.1GHz \\ \Omega_i &= 46MHz\end{aligned}$$

For low temperature, we do Taylor expansion
For large argument

$$Z(x) \xrightarrow[x\gg 1]{} i\sqrt{\pi}e^{-x^2} - \frac{1}{x}\left(1+\frac{1}{2x^2}+\cdots\right)$$

From Eq.(2.2.9)

$$\epsilon(\vec{k},\omega) = 1+\sum_s \frac{2\omega_{ps}^2}{k^2V_s^2}\sum_n e^{-b}I_n(b)\left[1+\frac{\omega}{k_zV_s}Z_n(\zeta_n)\right] \qquad (2.2.1.1)$$

where

$$V_s = \sqrt{\frac{2T_s}{m_s}}$$

Assume that $\epsilon(\vec{k},\omega) = 1 + \chi_e(\vec{k},\omega) + \chi_i(\vec{k},\omega)$

$$\chi_e(\vec{k},\omega) = \frac{2\omega_{pe}^2}{k^2 V_e^2} \sum_{n=-\infty}^{\infty} e^{-b_e} I_n(b_e) \left[1 + \frac{\omega}{k_z V_e} Z_n(\zeta_{ne})\right] \qquad (2.2.1.2)$$

$$\chi_i(\vec{k},\omega) = \frac{2\omega_{pi}^2}{k^2 v_i^2} \sum_{n=-\infty}^{\infty} e^{-b_i} I_n(b_i) \left[1 + \frac{\omega}{k_z v_i} Z_n(\zeta_{ni})\right] \qquad (2.2.1.3)$$

$$\text{where} \quad b_e = \frac{k_\perp^2 V_e^2}{2\Omega_e^2}, \quad b_i = \frac{k_\perp^2 v_i^2}{2\Omega_i^2}$$

$$\zeta_{ne} = \frac{\omega + n\Omega_e}{k_z V_e}, \quad \zeta_{ni} = \frac{\omega - n\Omega_i}{k_z v_i}$$

$$\Omega_e = \left|\frac{q_e B_0}{m_e}\right|, \quad \Omega_i = \frac{q_i B_0}{m_i} > 0$$

$$\begin{aligned}\chi_e(\vec{k},\omega) &= \frac{2\omega_{pe}^2}{k^2 v_e^2} e^{-b_e} \left\{ I_1(b_e) \left[2 + \frac{\omega}{k_z V_e}\left(Z_1(\zeta_{1e}) + Z_{-1}(\zeta_{-1e})\right)\right]\right. \\ &\quad + I_0(b_e)\left[1 + \frac{\omega}{k_z v_3} Z_0(\zeta_{0e})\right] \\ &\quad + \sum_{n=2}^{\infty} I_n(b_e)\left[2 + \frac{\omega}{k_z V_e}\left(Z_n(\zeta_{ne}) + Z_{-n}(\zeta_{-ne})\right)\right] \qquad (2.2.1.4)\end{aligned}$$

Here we used $I_n(b_e) = I_{-n}(b_e)$

Since ζ_{ne} is large, we use the asymptotic expansion of the dispersion function.

$$\begin{aligned}Z_1(\zeta_{1e}) + Z_{-1}(\zeta_{-1e}) &\simeq i\sqrt{\pi}\left(e^{-\zeta_{1e}^2} + e^{-\zeta_{-1e}^2}\right) - \frac{1}{\zeta_{1e}} - \frac{1}{2\zeta_{1e}^3} - \frac{1}{\zeta_{-1e}} - \frac{1}{2\zeta_{-1e}^3} \\ &= i\sqrt{\pi}\left[\exp\left(-\frac{(\omega+\Omega_e)^2}{k_z^2 v_e^2}\right) + \exp\left(-\frac{(\omega-\Omega_e)^2}{k_z^2 v_e^2}\right)\right] \\ &\quad - \left(\frac{k_z V_e}{\omega+\Omega_e} + \frac{k_z V_e}{\omega-\Omega_e} + \frac{1}{2}\,\frac{k_z^3 v_e^3}{(\omega+\Omega_e)^3} + \frac{1}{2}\,\frac{k_z^3 v_e^3}{(\omega-\Omega_e)^3}\right) \\ &= i\sqrt{\pi}\left[\exp\left(-\frac{(\omega+\Omega_e)^2}{k_z^2 v_e^2}\right) + \exp\left(-\frac{(\omega-\Omega_e)^2}{k_z^2 v_e^2}\right)\right] \\ &\quad - k_z V_e \frac{2\omega}{\omega^2 - \Omega^2} - \frac{1}{2}\, k_z^3 v_e^3\, \frac{2\omega(\omega^2 + 3\Omega^2)}{(\omega^2 - \Omega^2)^3}\end{aligned}$$

$$\begin{aligned}2 \;&+\; \frac{\omega}{k_z V_e}\left(Z_1(\zeta_{1e}) + Z_{-1}(\zeta_{1e})\right) \\ &= i\sqrt{\pi}\frac{\omega}{k_z V_e}\left[e^{-\zeta_{1e}^2} + e^{-\zeta_{-1e}^2}\right] + 2 - \frac{2\omega^2}{\omega^2 - \Omega_e^2} - \frac{1}{2}k_z^2 V_e^2 \frac{2\omega^2(\omega^2 + 3\Omega_e^2)}{(\omega^2 - \Omega_e^2)^3} \\ &= i\sqrt{\pi}\frac{\omega}{k_z V_e}\left[e^{-\frac{(\omega+\Omega_e)^2}{k_z^2 V_e}} + e^{-\frac{(\omega+-\Omega_e)^2}{k_z^2 V_e}}\right] - \frac{2\Omega^2}{\omega^2 - \Omega_e^2} - k_z^2 V_e^2 \frac{\omega^2(\omega^2 + 3\Omega_e^2)}{(\omega^2 - \Omega_e^2)^3} \qquad (2.2.1.5)\end{aligned}$$

$$\begin{aligned}
Z_n(\zeta_{ne}) + Z_{-n}(\zeta_{-ne}) &\simeq i\sqrt{\pi}\left(e^{-\zeta_{ne}^2} + e^{-\zeta_{-ne}^2}\right) - \frac{1}{\zeta_{ne}} - \frac{1}{\zeta_{-ne}} \\
&= i\sqrt{\pi}\left(e^{-\zeta_{ne}^2} + e^{-\zeta_{-ne}^2}\right) - k_z V_e \frac{2\omega}{\omega^2 - n^2\Omega_e^2} \\
&\simeq -2k_z V_e \ \frac{\omega}{\omega^2 - n^2\Omega_e^2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 2 + \frac{\omega}{k_z V_e}\left(Z_n(\zeta_{ne}) + Z_{-n}(\zeta_{-ne})\right) &= 2 - \frac{\omega}{k_z V_e}\ 2k_z V_e \ \frac{\omega}{\omega^2 - n^2\Omega_e^2} \\
&= \frac{-2n^2\Omega_e^2}{\omega^2 - n^2\Omega_e^2} \qquad (2.2.1.6)
\end{aligned}$$

$$\begin{aligned}
Z_0(\zeta_{0e}) &\simeq i\sqrt{\pi}e^{-\zeta_{0e}^2} - \frac{1}{\zeta_{0e}} - \frac{1}{2\zeta_{0e}^3} \\
&= i\sqrt{\pi}e^{1\frac{\omega^2}{k_z^2 V_e^2}} - \frac{k_z V_e}{\omega} - \frac{1}{2}\frac{k_z^3 V_e^3}{\omega^3}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 1 + \frac{\omega}{k_z V_e} Z_0(\zeta_{0e}) &\simeq i\sqrt{\pi}\frac{\omega}{k_z V_e}e^{-\frac{\omega^2}{k_z^2 V_e^2}} + 1 - \frac{\omega}{k_z V_e}\frac{k_z V_e}{\omega} - \frac{1}{2}\frac{\omega}{k_z V_e}\frac{k_z^3 V_e^3}{\omega^3} \\
&= i\sqrt{\pi}\frac{\omega}{k_z V_e}e^{-\frac{\omega^2}{k_z^2 V_e^2}} - \frac{k_z^2 V_e^2 e}{2\omega^2} \qquad (2.2.1.7)
\end{aligned}$$

Substitution of Eqs. (2.2.1.5)-(2.2.1.7) into Eq. (2.2.1.4)
Then,

$$\begin{aligned}
\mathrm{Re}(\chi_e) &= \frac{2\omega_{pe}^2}{k^2 V_e^2} e^{-b_e}\left\{ I_1(b_e)\left[-\frac{2\Omega_e^2}{\omega^2 - \Omega_e^2} - k_z^2 V_e^2 \frac{\omega^2(\omega^2 + 3\Omega_e^2)}{(\omega^2 - \Omega_e^2)^3} \right] \right. \\
&+ I_0(b_e)\left[-\frac{k_z^2 V_e^2}{2\omega^2} \right] \\
&+ \left. \sum_{n=2}^{\infty} I_n(b_e)\left[\frac{-2n^2\Omega^2}{\omega^2 - n^2\Omega_e^2} \right] \right\}
\end{aligned}$$

For low temperature, i.e., $b_e \ll 1$

$$\begin{aligned}
e^{-b_e} &\simeq 1 - b_e \\
I_1(b_e) &\simeq \frac{b}{2} \\
I_0(b_e) &= 1
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}(\chi_e) &= \frac{2\omega_{pe}^2}{k^2 V_e^2}(1 - b_e)\left[\frac{-b_e}{2}\left(\frac{2\Omega_e^2}{\omega^2 - \Omega_e^2} + k_z^2 V_e^2 \frac{\omega^2(\omega^2 + 3\Omega_e^2)}{(\omega^2 - \Omega_e^2)^3}\right) - \frac{k_e^2 V_e^2}{2\omega^2} \right] \\
&- \frac{2\omega_{pe}^2}{k^2 V_e^2}\sum_{n=2}^{\infty} e^{-b_e} I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2 - n^2\Omega_e^2}
\end{aligned}$$

We take first-order term of b_e. Therefore $\epsilon_{R,e}$ is

$$\begin{aligned}
\mathrm{Re}(\chi_e) &= \frac{2\omega_{pe}^2}{k^2V_e^2}\left(-\frac{b_e}{2}\right)\left[\frac{2\Omega_e^2}{\omega^2-\Omega_e^2}+k_z^2V_e^2\frac{\omega^2(\omega^2+3\Omega_e^2)}{(\omega^2-\Omega_e^2)^3}\right]-\frac{2\omega_{pc}^2}{k^2V_e^2}\frac{k_z^2V_e^2}{2\omega^2} \\
&\quad+\frac{2\omega_{pe}^2}{k^2V_e^2}(b_e)\frac{k_z^2V_e^2}{2\omega^2}-\frac{2\omega_{pe}^2}{k^2V_e^2}\sum_{n=2}^{\infty}e^{-b_e}I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2-n^2\Omega_e^2} \\
&= \frac{\omega pe^2}{k^2V_e^2}\frac{k_\perp^2V_e^2}{2\Omega_e^2}\frac{2\Omega_e^2}{\omega^2-\Omega_e^2}-\frac{\omega pe^2}{k^2V_e^2}\frac{k_\perp^2V_e^2}{2\Omega_e^2}k_z^2V_e^2\frac{\omega^2(\omega^2+3\Omega_e^2)}{(\omega^2-\Omega_e^2)^3}-\frac{2\omega pe^2}{k^2V_e^2}\frac{k_z^2V_e^2}{2\omega_e^2} \\
&\quad+\frac{2\omega pe^2}{k^2V_e^2}\frac{k_\perp^2V_e^2}{2\Omega_e^2}\frac{k_z^2V_e^2}{2\omega_e^2}-\frac{2\omega pe^2}{k^2V_e^2}\sum_{n=2}^{\infty}e^{-b_e}I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2-n^2\Omega_e^2} \\
&= -\frac{\omega pe^2}{\omega^2-\Omega_e^2}\frac{k_\perp^2}{k^2}-\frac{\omega pe^2}{2\Omega_e^2}\frac{k_\perp^2}{k^2}\frac{k_z^2V_e^2}{\omega^2}\frac{\omega^4(\omega^2+3\Omega_e^2)}{(\omega^2-\Omega_e^2)^3}-\frac{\omega pe^2}{\omega_e^2}\frac{k_z^2}{k^2} \\
&\quad+\frac{\omega pe^2}{2\Omega_e^2}\frac{k_\perp^2}{k^2}\frac{k_z^2V_e^2}{\omega^2}-\frac{2\omega pe^2}{k^2V_e^2}\sum_{n=2}^{\infty}e^{-b_e}I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2-n^2\Omega_e^2} \\
&= -\frac{\omega pe^2}{\omega_e^2}\frac{k_z^2}{k^2}-\frac{\omega pe^2}{\omega^2-\Omega_e^2}\frac{k_\perp^2}{k^2}+\epsilon_{te} \qquad (2.2.1.8)
\end{aligned}$$

where

$$\epsilon_{te}=\frac{\omega pe^2k_z^2}{2\omega^2k^2}\left[\frac{k_\perp^2V_e^2}{\Omega_e^2}-\frac{k_\perp^2V_e^2}{\Omega_e^2}\frac{\omega^4(\omega^2+3\Omega_e^2)}{(\omega^2-\Omega_e^2)^3}\right]-\frac{1}{k^2\lambda_{De}^2}\sum_{n=2}^{\infty}e^{-b_e}I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2-n^2\Omega_e^2}$$

$$\begin{aligned}
\mathrm{Im}(\chi_e) &= \frac{2\omega_{pe}^2}{k^2V_e^2}e^{-b_e}\left\{I_1(b_e)\left[e^{-\frac{(\omega+\Omega_e)^2}{k_z{}^2V_e^2}}+e^{-\frac{(\omega-\Omega_e)^2}{k_z{}^2V_e^2}}\right]+I_0(b_e)e^{-\frac{\omega^2}{k_z{}^2V_e^2}}\right\} \\
&\simeq \sqrt{\pi}\frac{2\omega_{pe}^2}{k^2V_e^2}\frac{\omega}{k_zV_e}(1-b_e)\left\{\frac{b_e}{2}\left[e^{-\frac{(\omega+\Omega_e)^2}{k_z^2V_e^2}}+e^{-\frac{(\omega-\Omega_e)^2}{k_z^2V_e^2}}\right]+e^{-\frac{\omega^2}{k_z^2V_e^2}}\right\} \\
&\simeq \sqrt{\pi}\frac{2\omega_{pe}^2}{k^2V_e^2}\frac{\omega}{k_zV_e}\frac{b_e}{2}\left[e^{-\frac{(\omega+\Omega_e)^2}{k_z^2V_e^2}}+e^{-\frac{(\omega-\Omega_e)^2}{k_z^2V_e^2}}\right] \\
&\quad+\sqrt{\pi}\frac{2\omega_{pe}^2}{k^2V_e^2}\frac{\omega}{k_zV_e}e^{-\frac{\omega^2}{k_z^2V_e^2}}-\sqrt{\pi}\frac{2\omega_{pe}^2}{k^2V_e^2}\frac{\omega}{k_zV_e}b_ee^{-\frac{\omega^2}{k_z^2V_e^2}} \\
&= \sqrt{\pi}\left\{\frac{2\omega_{pe}^2}{2\Omega_e^2}\frac{k_\perp^2}{k^2}\frac{\omega}{k_zV_e}\left[e^{-\frac{(\omega+\Omega_e)^2}{k_z^2V_e^2}}+e^{-\frac{(\omega-\Omega_e)^2}{k_z^2V_e^2}}\right]\right. \\
&\quad\left.+\frac{1}{k^2\lambda_{De}^2}\frac{\omega}{k_zV_e}\left[1-\frac{k_\perp^2V_e^2}{2\Omega_e^2}\right]e^{-\frac{\omega^2}{k_z^2V_e^2}}\right\} \qquad (2.2.1.9)
\end{aligned}$$

where $\lambda_{De}^{-2}=\frac{2\omega_{pe}^2}{V_e^2}$

Similarly, we get $\chi_i(\vec{k},\omega)$ easily.

$$\begin{aligned}
\omega_{pe} &\longrightarrow \omega_{pi} \\
\Omega_e &\longrightarrow \Omega_i \\
V_e &\longrightarrow V_i = \sqrt{\frac{2T_i}{m_i}} \\
\lambda_{De}^{-2} &\longrightarrow \lambda_{Di}^{-2} = \frac{2\omega_{pi}^2}{V_i^2} \\
b_e &\longrightarrow b_i = \frac{k_\perp^2 V_i^2}{2\Omega_i^2}
\end{aligned}$$

$$\mathrm{Re}\left(\chi_i(\vec{k},\omega)\right) = -\frac{\omega_{pi}^2}{\omega^2}\frac{k_z^2}{k^2} - \frac{\omega_{pi}^2}{\omega^2-\Omega_i^2}\frac{k_\perp^2}{k^2} + \epsilon_{ti} \tag{2.2.1.10}$$

where

$$\begin{aligned}
\epsilon_{ti} =& \frac{\omega_{pi}^2 k_z^2}{2\omega^2 k^2}\left[\frac{k_\perp^2 V_i^2}{\Omega_i^2} - \frac{k_\perp^2 V_i^2}{\Omega_i^2}\frac{\omega^4(\omega^2+3\Omega_i^2)}{(\omega^2-\Omega_i^2)^3}\right] \\
&- \frac{1}{k^2\lambda_{Di}^2}\sum_{n=2}^{\infty} e^{-b_i} I_n(b_i)\frac{2n^2\Omega_i^2}{\omega^2-n^2\Omega_i^2}
\end{aligned} \tag{2.2.1.11}$$

$$\begin{aligned}
\mathrm{Im}(\chi_i) =& \sqrt{\pi}\left[\frac{\omega_{pi}^2}{2\Omega_i^2}\frac{k_\perp^2}{k^2}\frac{\omega}{k_z V_i}\left\{e^{-\frac{(\omega+\Omega_i)^2}{k_z^2 V_i^2}} + e^{-\frac{(\omega-\Omega_i)^2}{k_z^2 V_i^2}}\right\}\right. \\
&\left. + \frac{1}{k^2\lambda_{Di}^2}\frac{\omega}{k_z V_i}\left(1-\frac{k_\perp^2 V_i^2}{2\Omega_i^2}\right)e^{-\frac{\omega^2}{k_z^2 V_i^2}}\right]
\end{aligned} \tag{2.2.1.12}$$

For getting electron modes with $k_z \neq 0$, one may neglect $\chi_i(\vec{k},\omega)$

$$
\begin{aligned}
\therefore \quad \epsilon(\vec{k},\omega) &= 1+\chi_e(\vec{k},\omega)=\epsilon_{R,e}+i\epsilon_{I,e} \\
\epsilon_{R,e} &= 1-\frac{\omega_{pe}^2}{\omega^2}\frac{k_z^2}{k^2}-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}+\frac{k_\perp^2}{k^2}+\epsilon_{te} \qquad (2.2.1.13) \\
\epsilon_{I,e} &= \sqrt{\pi}\left[\frac{\omega_{pe}^2}{2\Omega_e^2}\frac{k_\perp^2}{k^2}\frac{\omega}{k_zV_e}\left(e^{-\frac{(\omega+\Omega_e)^2}{k_z^2V_e^2}}+e^{-\frac{(\omega-\Omega_e)^2}{k_z^2V_e^2}}\right)\right. \\
&\quad \left.+\frac{1}{k^2\lambda_{De}^2}\frac{\omega}{k_zV_e}\left(1-\frac{k_\perp^2V_e^2}{2\Omega_e^2}\right)e^{-\frac{\omega^2}{k_z^2V_e^2}}\right] \qquad (2.2.1.14) \\
\epsilon_{te} &= \frac{\omega_{pe}^2k_z^2}{2\omega^2k^2}\left[\frac{k_\perp^2V_e^2}{\Omega_e^2}-\frac{k_\perp^2V_e^2}{\Omega_e^2}\frac{\omega^4(\omega^2+3\Omega_e^2)}{(\omega^2-\Omega_e^2)^3}\right] \\
&\quad -\frac{1}{k^2\lambda_{De}^2}\sum_{n=2}^{\infty}e^{-b_e}I_n(b_e)\frac{2n^2\Omega_e^2}{\omega^2-n^2\Omega_e^2} \qquad (2.2.1.15)
\end{aligned}
$$

$\epsilon_{R,e}=\epsilon_{R,e}(\vec{k},\omega)=0$ gives

$$
1-\frac{\omega_{pe}^2}{\omega^2}\frac{k_z^2}{k^2}-\frac{\omega_{pe}^2}{\omega^2-\Omega_e^2}\frac{k_\perp^2}{k^2}=0 \qquad (2.2.1.16)
$$

$$
\begin{aligned}
&\Rightarrow\; k^2\omega^2(\omega^2-\Omega_e^2)-k_z^2\omega_{pe}^2(\omega^2-\Omega_e^2)-k_\perp^2\omega_{pe}^2\omega^2=0 \\
&\Rightarrow\; k^2\omega^4-k^2\Omega_e^2\omega^2-k_z^2\omega_{pe}^2\omega^2-k_\perp^2\omega_{pe}^2\omega^2+k_z^2\omega_{pe}^2\Omega_e^2=0 \\
&\Rightarrow\; k^2\omega^4-(k^2\Omega_e^2+(k_z^2+k_\perp^2)\omega_{pe}^2)\omega^2+k_z^2\omega_{pe}^2\Omega_e^2=0 \\
&\Rightarrow\; k^2\omega^4-k^2(\Omega_e^2+\omega_{pe}^2)\omega^2+k_z^2\omega_{pe}^2\Omega_e^2=0 \\
&\Rightarrow\; k^2\omega^4-k^2\omega_{UH}^2\omega^2+k_z^2\omega_{pe}^2\Omega_e^2=0
\end{aligned}
$$

where, $\omega_{UH}^2=\omega_{pe}^2+\Omega_e^2$

$$
\begin{aligned}
\therefore \quad \omega^2 &= \frac{k^2\omega_{UH}^2\pm\sqrt{k^4\omega_{UH}^4-4k^2k_z^2\omega_{pe}^2\Omega_e^2}}{2k^2} \\
&= \frac{1}{2}\left[\,\omega_{UH}^2\pm(\omega_{UH}^2-4\Omega_e^2\omega_{pe}^2k_z^2/k^2)^{1/2}\,\right] \qquad (2.2.1.17)
\end{aligned}
$$

For $k_\perp\to\infty$, $\omega^2=\frac{1}{2}(\omega_{UH}^2\pm\omega_{UH}^2)=\omega_{UH}^2$ (Upper Hybrid Resonance)

- Plots of Eq. (2.2.1.17)

  The plots of Eq. (2.2.1.17) are inserted using Mathematica program.

- Lists of plots

  1. Plot 3D: ω vs. (k_x,k_z)

2. Contour plots of (1)
3. Plot 2D: ω vs. (k_x, k_z)

<u>B. Electron Bernstein Waves</u> ($k_z \to 0,\ \omega = |n\Omega_e|$)
In the limit $k_z \to 0 (k_z \ll k_\perp)$, $\epsilon = 0$ gives electron Bernstein modes at $\omega \simeq |n\Omega_e|$.

$$\begin{aligned}\epsilon(\vec{k}, \omega) &= 1 + \sum_s \frac{2\omega_{ps}^2}{k^2 V_s^2} \sum_n e^{-b} I_n(b)[1 + \frac{\omega}{k_z V_s} Z_n(\zeta_n)] \\ &= 1 + \frac{2\omega_{pe}^2}{k^2 V_e^2} \sum_{n=-\infty}^{\infty} e^{-b_e} I_n(b_e)[1 + \frac{\omega}{k_z V_e} Z_n(\zeta_{ne})]\end{aligned}$$

where $\zeta_{ne} = \frac{\omega + n\Omega_e}{k_z V_e}$ and $\Omega_e = |\frac{q_e B_0}{m_e}| > 0$

In the limit $k_z \to 0$, the damping terms (imaginary parts) disappear except precisely at $\omega = |n\Omega_e|$.

i) For $n = 0$

$$\begin{aligned}\epsilon(\vec{k}, \omega) &\simeq 1 + \frac{2\omega_{pe}^2}{k^2 V_e^2} e^{-b_e} I_0(b_e) \left[1 + \frac{\omega}{k_z V_e}\left(i\sqrt{\pi} e^{-\frac{\omega^2}{k_z^2 v_e^2}} - \frac{k_z V_e}{\omega}\right)\right] \\ &\simeq 1 + \frac{2\omega_{pe}^2}{k^2 v_e^2} e^{-b_e} I_0(b_e) \left[1 - \frac{\omega}{k_z V_e}\frac{k_z V_e}{\omega}\right] = 1 + 0\end{aligned}$$

No contribution in summation

ii) For $n \neq 0$

$$\epsilon(\vec{k}, \omega) = 1 + \frac{2\omega_{pe}^2}{k^2 V_e^2} \sum_{n=1}^{\infty} e^{-b_e} I_n(b_e)[2 + \frac{\omega}{k_z V_e}(Z_n(\zeta_{ne}) + Z_n(\zeta_{ne}))]$$

$$\begin{aligned}& 2 + \frac{\omega}{k_z V_e}(Z_n(\zeta_{ne} + Z_{-n}(\zeta_{-ne})) \\ \simeq\ & 2 + \frac{\omega}{k_z V_e}[i\sqrt{\pi} e^{-\zeta_{ne}^2} - \frac{1}{\zeta_{ne}} + i\sqrt{\pi} e^{-\zeta_{-ne}^2} - \frac{1}{\zeta_{-ne}}] \\ \simeq\ & 2 + \frac{\omega}{k_z V_e}(i\sqrt{\pi})(e^{-\zeta_{ne}^2} + e^{-\zeta_{-ne}^2}) - \frac{\omega}{k_z V_e}\left[\frac{k_z V_e}{\omega + n\Omega_e} + \frac{k_z V_e}{\omega - n\Omega_e}\right] \\ \simeq\ & 2 - \omega\left[\frac{1}{\omega + n\Omega_e} + \frac{1}{\omega - n\Omega_e}\right] \\ =\ & 2 - \omega\frac{2\omega}{\omega^2 - n^2\Omega_e^2} \\ =\ & \frac{-2n^2\Omega_e^2}{\omega^2 - n^2\Omega_e^2}\end{aligned}$$

Here, the imaginary terms is neglected if one may preserve $\omega \rightarrow \omega + i\nu$, $\nu > 0$. And, we just took the first term in its asymptotic expansion, $Z_n(\zeta_{ne}) \simeq -\frac{1}{\zeta}$.

Thus, for $k_z \rightarrow 0$

$$
\begin{aligned}
\epsilon(\vec{k},\omega) &= 1 - \frac{2\omega_{pe}^2}{k^2 V_e^2} \sum_{n=1}^{\infty} e^{-b_e} I_n(b_e) \frac{2n^2\Omega_e^2}{\omega^2 - n^2\Omega_e^2} \\
&= 1 - \frac{2\omega_{pe}^2}{k_\perp^2 V_e^2} \sum_{n=1}^{\infty} e^{-b_e} I_n(b_e) \frac{2n^2\Omega_e^2}{\omega^2 - n^2\Omega_e^2}
\end{aligned}
$$

Since, $b_e = \frac{k_\perp^2 V_e^2}{2\Omega_e^2}$,

$$
\begin{aligned}
\epsilon(\vec{k},\omega) &= 1 - \frac{\omega_{pe}^2}{\Omega_e^2} \frac{2}{b_e} \sum_{n=1}^{\infty} e^{-b_e} I_n(b_e) \frac{n^2}{(\omega/\Omega_e)^2 - n^2} \\
&= 1 - \frac{\omega_{pe}^2}{\Omega_e^2} \frac{\alpha(Q,b_e)}{b_e} = 0 \qquad (2.2.1.18)
\end{aligned}
$$

This is a "Dispersion Relation for E-Bernstein Waves".

Where,

$$\alpha(Q,b_e) = 2\sum_{n=1}^{\infty} e^{-b_e} I_n(b_e) \frac{n^2}{Q^2 - n^2} \qquad (2.2.1.19)$$

$$\text{and } \; Q = \frac{\omega}{\Omega_e} \qquad (1)$$

The solution of Eq (2.2.1.18) gives electron Bernstein waves.
The function $\alpha(Q,b_e)$ can be expressed an expansion in ascending power of b_e,

$$\alpha(Q,b_e) = \frac{b_e}{Q^2 - 1^2} + \frac{1\cdot 3 b_e^2}{(Q^2-1^2)(Q^2-2^2)} + \frac{1\cdot 3\cdot 5 b_e^3}{(Q^2-1^2)(Q^2-2^2)(Q^2-3^2)} + \dots$$

This expression shows that resonance at the nth cyclotron harmonic appear only when terms up to at least b_e^{n-1} are pertained in the dispersion relation.

The characteristics of $\alpha(Q,b_e)$ is shown in page 295-300 of T.H. Stix's book in detail.

It is convenient to put Eq(2.2.1.18) into the from

$$\frac{\Omega_e^2}{\omega_{pe}^2} = \frac{\alpha(Q, b_e)}{b_e} \tag{2.2.1.20}$$

The plots of Eq.(2.2.1.20): Q vs. $\sqrt{b_e}$ for several values of $\frac{\omega_{pe}^2}{\Omega_e^2}$

Since $Q = \frac{\omega}{\Omega_e}$ and $b_e = \frac{k_\perp^2 V_e^2}{2\Omega_e^2} = k_\perp^2 \rho_e^2 \quad (\rho_e^2 = \frac{V_e^2}{2\Omega_e^2})$

$$\sqrt{b_e} = k_\perp \rho_e$$

$$Q \text{ vs. } \sqrt{b_e} = k_\perp \rho_e \quad \Rightarrow \quad (\frac{\omega}{\Omega_e}) \text{ vs. } k_\perp \rho_e$$

Eq. (2.2.1.20) becomes

$$\begin{aligned} \alpha(Q, b_e) &= b_e \frac{1}{\omega_{pe}^2/\Omega_e^2} \\ \rightarrow \frac{\alpha(Q, b_e)}{b_e} &= \frac{1}{\omega_{pe}^2/\Omega_e^2} = \frac{1}{X} \end{aligned}$$

Left-hand Side :

$$\frac{\alpha(Q, b_e)}{b_e} = \frac{1}{Q^2 - 1^2} + \frac{1 \cdot 3 \cdot b_e}{(Q^2 - 1^2)(Q^2 - 2^2)} + \frac{1 \cdot 3 \cdot 5 \cdot b_e^2}{(Q^2 - 1^2)(Q^2 - 2^2)(Q^2 - 3^2)} + \cdots$$

Then, for some harmonics,

$$\begin{aligned} (1)\ n = 2: \quad \frac{\alpha(Q, b_e)}{b_e} &= \frac{1}{Q^2 - 1^2} + \frac{1 \cdot 3 \cdot b_e}{(Q^2 - 1^2)(Q^2 - 2^2)} = \frac{1}{X} \\ (2)\ n = 3: \quad \frac{\alpha(Q, b_e)}{b_e} &= \frac{1}{Q^2 - 1^2} + \frac{1 \cdot 3 \cdot b_e}{(Q^2 - 1^2)(Q^2 - 2^2)} \\ &\quad + \frac{1 \cdot 3 \cdot 5 \cdot b_e^2}{(Q^2 - 1^2)(Q^2 - 2^2)(Q^2 - 3^2)} \\ &= \frac{1}{X} \\ (3)\ n = 4: \quad \frac{\alpha(Q, b_e)}{b_e} &= \frac{1}{Q^2 - 1^2} + \frac{1 \cdot 3 \cdot b_e}{(Q^2 - 1^2)(Q^2 - 2^2)} \\ &\quad + \frac{1 \cdot 3 \cdot 5 \cdot b_e^2}{(Q^2 - 1^2)(Q^2 - 2^2)(Q^2 - 3^2)} \\ &\quad + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot b_e^3}{(Q^2 - 1^2)(Q^2 - 2^2)(Q^2 - 3^2)(Q^2 - 4^2)} \\ &= \frac{1}{X} \end{aligned}$$