1 Dispersion Relation in a Cold Uniform Plasmas

As long as $T_e = T_i = 0$, the waves described can easily be generalized to an arbitrary number of charged particle species and an arbitrary angle of propagation θ relative to the magnetic field. Waves that depend on finite T, such as ion acoustic waves, are not included in this treatment.
The fourth Maxwell equation:

$$\nabla \times \vec{B} = \mu_0(\vec{j} + \epsilon_0 \dot{\vec{E}})$$

where $\vec{j}$ is the plasma current due to the motion of the various charged particle species s, with density n_s, charge q_s, and velocity v_s:

$$\vec{j} = \sum_s n_s q_s v_s$$

Considering the plasma to be a dielectric with internal currents $\vec{j}$,

$$\nabla \times \vec{B} = \mu_0 \dot{\vec{D}}$$

where

$$\vec{D} = \epsilon_0 \vec{E} + \frac{i}{\omega} \vec{j}$$

Here we have assumed an exp(-iωt) dependence for all plasma motions.
A conductive tensor $\overleftrightarrow{\sigma}$ (because of the magnetic field $B_0 \hat{z}$);

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E}$$

Thus,

$$\vec{D} = \epsilon_0(\overleftrightarrow{1} + \frac{i}{\epsilon\omega} \overleftrightarrow{\sigma}) \cdot \vec{E} = \overleftrightarrow{\epsilon} \cdot \vec{E}$$

The effective dielectric tensor of the plasma:

$$\overleftrightarrow{\epsilon} = \epsilon_0(\overleftrightarrow{1} + \frac{i}{\epsilon\omega} \overleftrightarrow{\sigma})$$

where $\overleftrightarrow{1}$ is the unit tensor.

To evaluate $\overleftrightarrow{\sigma}$, we use the "linearized fluid equation" of motion for species s, neglecting the collision and pressure terms:

$$m_s \frac{\partial \vec{v_s}}{\partial t} = q_s(\vec{E} + \vec{v_s} \times \vec{B_0})$$

Defining the cyclotron and plasma frequencies for each species as

$$\Omega_s \equiv |\frac{q_s B_0}{m_s}| \qquad \omega_{ps} \equiv \frac{n_0 q_s^2}{\epsilon_0 m_s}$$

Note that $\Omega_s > 0$ hereafter.

We can separate "linearized fluid equation" into x, y, and z components and solve for v_s, obtaining

$$\begin{aligned} v_{xs} &= \frac{iq_s}{m_s\omega}\frac{[E_x \pm i(\Omega_s/\omega)E_y]}{1-(\Omega_s/\omega)^2} \\ v_{ys} &= \frac{iq_s}{m_s\omega}\frac{[E_y \pm i(\Omega_s/\omega)E_x]}{1-(\Omega_s/\omega)^2} \\ v_{zs} &= \frac{iq_s}{m_s\omega}E_z \end{aligned}$$

where $\pm$ stands for the sign of q_s. The plasma current is

$$\vec{j} = \sum_s n_{0s} q_s \vec{v_s}$$

so that

$$\begin{aligned} \frac{i}{\epsilon_0\omega}j_x &= \sum_i \frac{in_{0s}}{\epsilon_0\omega}\frac{iq_s^2}{m_s\omega}\frac{[E_x \pm i(\Omega_s/\omega)E_y]}{1-(\Omega_s/\omega)^2} \\ &= \sum_i \frac{\omega_{ps}^2}{\omega^2}\frac{[E_x \pm i(\Omega_s/\omega)E_y]}{1-(\Omega_s/\omega)^2} \end{aligned}$$

Using the identities

$$\begin{aligned} \frac{1}{1-(\Omega_s/\omega)^2} &= \frac{1}{2}[\frac{\omega}{\omega\mp\Omega_s}+\frac{\omega}{\omega\pm\Omega_s}] \\ \pm\frac{\Omega_s/\omega}{1-(\Omega_s/\omega)^2} &= \frac{1}{2}[\frac{\omega}{\omega\mp\Omega_s}-\frac{\omega}{\omega\pm\Omega_s}], \end{aligned}$$

$$\frac{1}{\epsilon_0\omega}j_x = -\frac{1}{2}\sum_s \frac{\omega_{ps}^2}{\omega^2}[(\frac{\omega}{\omega\mp\Omega_s}+\frac{\omega}{\omega\pm\Omega_s})E_x + (\frac{\omega}{\omega\mp\Omega_s}-\frac{\omega}{\omega\pm\Omega_s})iE_y]$$

Similarly, the y and z components are

$$\frac{1}{\epsilon_0\omega}j_y = -\frac{1}{2}\sum_s \frac{\omega_{ps}^2}{\omega^2}[(\frac{\omega}{\omega\pm\Omega_s}-\frac{\omega}{\omega\mp\Omega_s})iE_x + (\frac{\omega}{\omega\mp\Omega_s}+\frac{\omega}{\omega\pm\Omega_s})E_y]$$

$$\frac{1}{\epsilon_0\omega}j_z = -\sum_s \frac{\omega_{ps}^2}{\omega^2}E_z$$

These give

$$\frac{1}{\epsilon_0}D_x = E_x - \frac{1}{2}\sum_s[\frac{\omega_{ps}^2}{\omega^2}(\frac{\omega}{\omega\mp\Omega_s}+\frac{\omega}{\omega\pm\Omega_s})E_x + \frac{\omega_{ps}^2}{\omega^2}(\frac{\omega}{\omega\mp\Omega_s}-\frac{\omega}{\omega\pm\Omega_s})iE_y]$$

Similarly with the y and z components, we obtain

$$\begin{aligned} \epsilon_0^{-1}D_x &= SE_x - iDE_y \\ \epsilon_0^{-1}D_y &= iDE_x + iSE_y \\ \epsilon_0^{-1}D_z &= PE_z \end{aligned}$$

Where

$$\begin{aligned} R &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}(\frac{\omega}{\omega \pm \Omega_s}) \\ L &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}(\frac{\omega}{\omega \mp \Omega_s}) \\ S &\equiv \frac{1}{2}(R+L) \quad D \equiv \frac{1}{2}(R-L) \\ P &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}. \end{aligned}$$

Or,

$$\begin{aligned} S &= 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} \\ D &= \sum_s \frac{\omega_{ps}^2}{\omega} \frac{\pm\Omega_s}{\omega^2 - \Omega_s^2} \\ P &= 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}. \end{aligned}$$

From $\vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E}$,

$$\overleftrightarrow{\epsilon} = \epsilon_0 \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \equiv \epsilon_0 \overleftrightarrow{K}$$

The wave equation by taking the curl of the equation $\nabla \times \vec{E} = -\dot{\vec{B}}$ and substituting $\nabla \times \vec{B} = \mu_0 \overleftrightarrow{\epsilon} \cdot \dot{\vec{E}}$:

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \epsilon_0 (\overleftrightarrow{K} \cdot \ddot{\vec{E}}) = -\frac{1}{c^2} \overleftrightarrow{K} \cdot \ddot{\vec{E}}$$

Assuming an $\exp(i\vec{k} \cdot \vec{r})$ spatial dependence of $\vec{E}$ and defining a vector index of refraction

$$\vec{N} = \frac{c}{\omega}\vec{k},$$

the wave equation becomes

$$\vec{N} \times (\vec{N} \times \vec{E}) + \overleftrightarrow{K} \cdot \vec{E} = 0$$

The uniform plasma is isotropic in the x-y plane (i.e. $k_y = 0$).
If θ is the angle between $\vec{k}$ and $\vec{B}_0$ we then have

$$N_x = n \sin\theta \quad N_z = n \cos\theta \quad N_y = 0$$

Using the elements of $\overleftrightarrow{K}$,

$$\overleftrightarrow{M} \cdot \vec{E} \equiv \begin{pmatrix} S - N^2\cos^2\theta & -iD & N^2 \sin\theta\cos\theta \\ iD & S - N^2 & 0 \\ N^2 \sin\theta\cos\theta & 0 & P - N^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

From this it is clear that the E_x, E_y components are coupled to E_z only if one deviates from the principal angles $\theta = 0$, $90°$.

The above equation is a set of three simultaneous, homogeneous equations; the condition for the existence of a solution is that the determinant of $\overleftrightarrow{M}$ vanish: $\| \overleftrightarrow{M} \| = 0$.

That is,

$$A'N^4 - B'N^2 + C' = 0. \text{ “Cold Plasma Dispersion Relation”}$$

Where

$$\begin{aligned} A' &= S\sin^2\theta + P\cos^2\theta, \\ B' &= RL\sin^2\theta + PS(1+\cos^2\theta), \\ C' &= PRL \end{aligned}$$

We have used the identity $S^2 - D^2 = RL$.

The solution of dispersion relation:

$$N^2 = \frac{B' \pm F}{2A'},$$

with

$$F^2 = (RL - PS)^2 \sin^4\theta + 4P^2D^2\cos^2\theta.$$

Alternately, using the notation of $\vec{N} = N_\perp \hat{x} + N_\| \hat{z} = N\sin\theta\hat{x} + N\cos\theta\hat{z}$, the dispersion relation can be rewritten by

$$AN_\perp^4 + BN_\perp^2 + C = 0$$

Where

$$\begin{aligned} A &= S, \\ B &= -(S+P)(S - N_\|^2) + D^2, \\ C &= P[(S - N_\|^2)^2 - D^2] \end{aligned}$$

The solution of $N_\perp^2$:

$$N_\perp^2 = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2A}$$

- **For Electron Cyclotron Wave (EC-wave)**
  1. ***Low Field Side (LFS) launch:***
     a. **(+) sign : O-mode (or Fast Wave)**
     b. **(−) sign : X-mode (or Slow Wave)**
  2. ***High Field Side (HFS) launch:***
     a. **(+) sign : X-mode (or Slow Wave)**
     b. **(−) sign : O-mode (or Fast Wave)**

- **For Lower Hybrid Wave (LH-wave)**

  a. **(+) sign : Slow Wave**
  b. **(−) sign : Fast Wave**

The dispersion relation was put into another form by Åström and Allis:

Expanding in minors of the second column of $\overleftrightarrow{M}$, we then obtain
$(iD)^2(P-N^2\sin^2\theta)+(S-N^2)\times[(S-N^2\cos^2\theta)(P-N^2\sin^2\theta)-N^4\sin^2\theta\cos^2\theta]=0$
By replacing $\cos^2\theta$ by $1-\sin^2\theta$, we can solve for $\sin^2\theta$, obtaining

$$\sin^2\theta = \frac{-P(N^4-2SN^2+RL)}{N^4(S-P)+N^2(PS-RL)}$$

We have used the identity $S^2-D^2=RL$, too. Similarly,

$$\cos^2\theta = \frac{SN^4-(PS+PL)N^2+PRL}{N^4(S-P)+N^2(PS-RL)}$$

Dividing the last two equations, we obtain

$$\tan^2\theta = \frac{P(N^4-2SN^2+RL)}{SN^4-(PS+RL)N^2+PRL}$$

Since $2S=R+L$, the numerator and denominator can be factored to

$$\tan^2\theta = \frac{P(N^2-R)(N^2-L)}{(SN^2-RL)(N^2-P)}$$

- When $\theta=0°$,

  $P=0$ (Langmuir wave)
  $N^2=R$ (R-wave)
  $N^2=L$ (L-wave)

- When $\theta=90°$,

  $N^2=RL/S$ (extraordinary wave)
  $N^2=P$ (ordinary wave)

1.1 Resonances ($N\to\infty$)

We then have

$$\tan^2\theta_{res} = -P/S$$

θ_{res} is the resonance cone angle.

This shows that the resonance frequencies depend on angle θ.

- If $\theta=0°$,

  $P=0$: Plasma resonance
  $S=\infty\begin{cases} R=\infty & \text{Electron Cyclotron Resonance} \\ L=\infty & \text{Ion Cyclotron Resonance}\end{cases}$

- If $\theta=90°$,

  $P=\infty$: No occurrence for finite ω_p and ω
  $S=0$: Upper Hybrid Resonance (ω_{UH} frequency) and Lower Hybrid Resonance (ω_{LH} frequency)

1.2 Cut-offs ($N = 0$)

Let $N = 0$ in $\| \overleftrightarrow{M} \| = 0$ and again using $S^2 - D^2 = RL$,

$$PRL = 0 \text{ indepedent of } \theta$$

- $R = 0$ (ω_R cutoff frequency)
- $L = 0$ (ω_L cutoff frequency)
- $P = 0$ (resonance for longitudinal wave, a cutoff for transverse waves): this degeneracy is due to our neglect of thermal motions.

1.3 Polarization

From wave equation,

$$iDE_x + (S - N^2)E_y = 0$$

Thus the polarization in the plane perpendicular to B_0 is given by

$$\frac{iE_x}{E_y} = \frac{N^2 - S}{D}$$

a. At resonance ($N^2 = \infty$), "Linearly Polarized"

b. At cutoff ($N^2 = 0$; $R = 0$ or $L = 0$; thus $S = \pm D$),

$$\frac{iE_x}{E_y} = -\frac{S}{D} = \mp 1 : \quad \text{"Circularly Polarized"}$$

c. At $\theta = 0$ ($N^2 = R$ or $N^2 = L$)

- For $N^2 = R$

  $$\frac{iE_x}{E_y} = \frac{R - S}{D} = \frac{R - 1/2(R + L)}{1/2(R - L)} = 1 \ : \text{ a right-hand circular polarization}$$

- For $N^2 = L$

  $$\frac{iE_x}{E_y} = \frac{L - S}{D} = \frac{L - 1/2(R + L)}{1/2(R - L)} = -1 \ : \text{ a left-hand circular polarization}$$

1.4 CMA diagram

The information contained in the cold dispersion relation is summarized in the Clemmow-Mullaly-Allis (CMA) diagram as seen in Fig. 1. One further result, not in the diagram, can be obtained easily from this formulation.

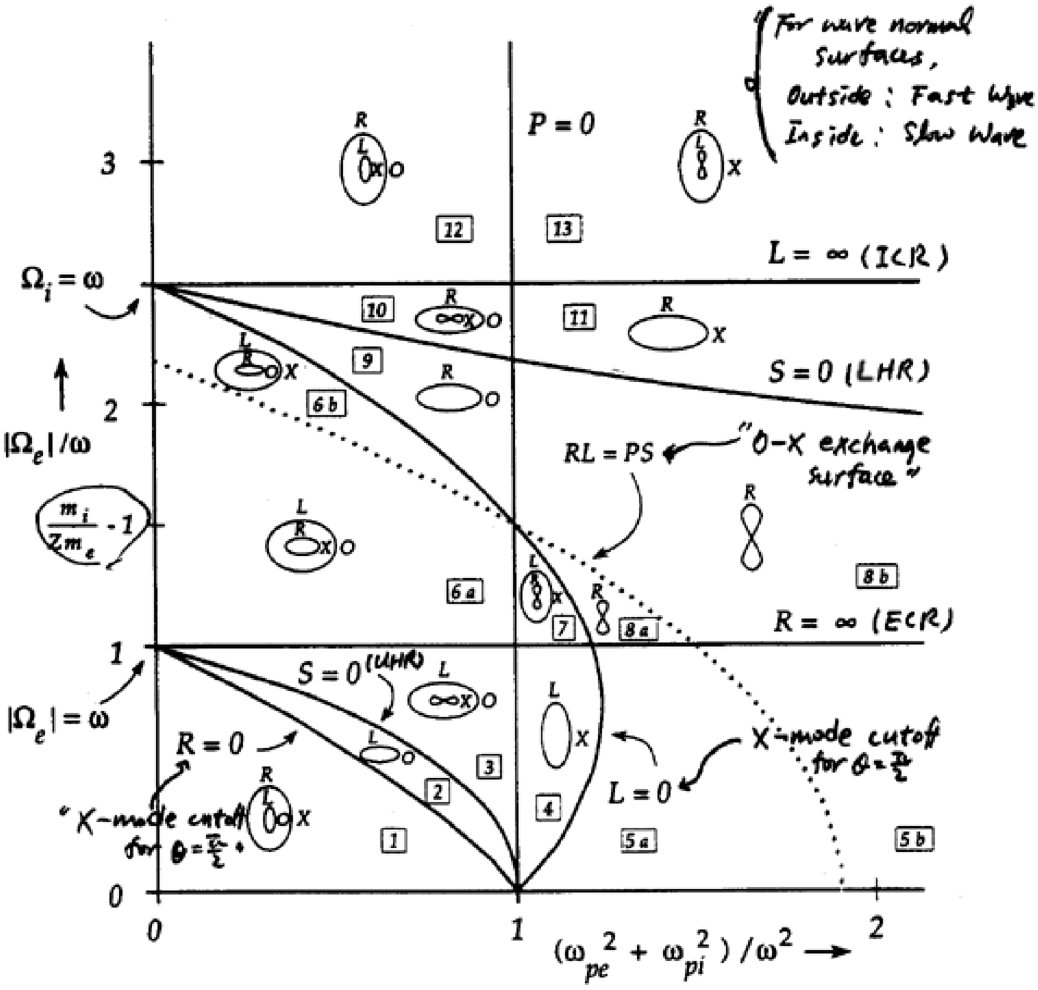

Figure 1: CMA diagram for a two-component plasma. The ion-to-electron mass ratio is chosen to be 2.5. Bounding surfaces appear as lines in this two-dimensional parameter space. Cross sections of wave-normal surfaces are sketched and labeled for each region. For these sketches the direction of the magnetic field is vertical. The small mass ratio can be misleading here: the L = 0 line intersects P = 0 at $\Omega_i/\omega = 1 - (Zm_e/m_i)$. *From T. Stix's book (AIP, 1992).*