C Quasi-linear Theory

- Velocity-space diffusion rate

$$D_v = \frac{\overline{(\nabla v)^2}}{2t} \propto E^2$$

where E is the amplitudes of the linear-theory modes
Squares of E → "Quasi-linear"

- Motivation of the development of quasi-linear theory.
  - Microinstabilities form an important part of wave theory.
  - And the questions arise, how will the mode amplitude grow?
  - What is the instability saturation mechanism?
  - Bibliography
    * W.E. Drummond and D.Pines (1961): "Nonlinear stability of Plasma Oscillations, General Atomic" GA-2386 (1961)
    * A.A.Vedenov, E.P.Velikhov and R.Z. Sagdeev (1961): "Nonlinear Oscillations of Rare field Plasma" Nuclear Fusion 1, 82 (1961)
    * Yu.A.Romanov and G.Filippov (1961): "The Interaction of Fast Electron Beams with Longitudinal Plasma Waves" sov.phys.-JETP 13, 87 (1961)
  - In which it was found that a temporally growing micro-instability acts back on the zero-order velocity distribution function.
  - Its effect on the distribution function is to produce velocity-space diffusion.
  - The diffusion tends to flatten $f_0(\vec{v})$ in this region and drive the instability growth rate to zero.
  - This saturation process can be viewed as a continuous diffusion through which the zero-order distribution function evolves slowly in time.
  - Two assumptions in quasi-linear theory.
    * The amplitudes of the perturbations in the plasma are not so large as to invalidate the use of zero-order orbits and of the spatially averaged distribution function, $f_0(\vec{v},t) =< f_0(\vec{r},\vec{v},t) >$
    * The effective wave spectrum should be sufficiently dense. Any appreciable coherence between modes will be destroyed by phase mixing.

- Electromagnetic Quasi-linear Theory

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}\vec{\nabla}_v \cdot \left(\vec{E} + \vec{v}\times\vec{B}\right) f = 0 \ \ (*f = f(\vec{r},\vec{v},t))$$

Averaging over a number of space and time periods of the rapid fluctuations. In addition, in presence of $\vec{B}_0$, we also average over the

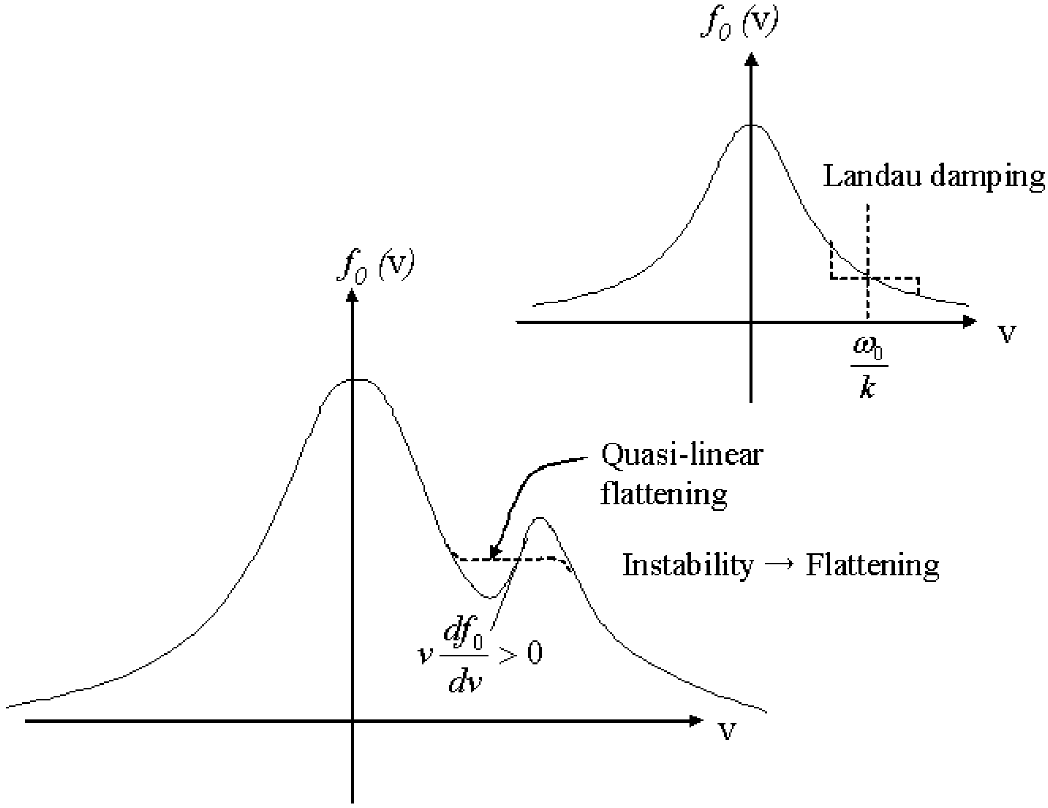

Figure 38: Velocity distribution for "bump-on-tail" instability. Real part of unstable frequencies are such that $v = \omega_0(k)/k$ lies in region where vdf_0/dv is positive (opposite sense to Landau damping). Quasi-linear diffusion due to these modes tends to flatten out the bump.

gyro-angle in velocity space:

$$\frac{\partial f_0(\vec{v},t)}{\partial t} = \left\langle \frac{\partial f}{\partial t} \right\rangle = -\left\langle \vec{v}\frac{\partial f}{\partial z} \right\rangle - \frac{q}{m}\left\langle \vec{\nabla}_v \cdot \left(\vec{E} + \vec{v}\times\vec{B}\right) f \right\rangle$$

slow evolution of $f_0(\vec{v},t) = \langle f(\vec{r},\vec{v},t)\rangle$

$$\begin{cases} \text{Space averaging } : \left\langle \frac{\partial f}{\partial z} \right\rangle = 0 \\ < E >= 0 \\ \text{Higher order contributions to } \left\langle \vec{\nabla}_v \cdot \left(\vec{E} + \vec{v}\times\vec{B}\right) f \right\rangle \text{ are neglected} \end{cases}$$

$$\begin{aligned} \frac{\partial f_0}{\partial t} &\simeq -\frac{q}{m}\left\langle \int_0^{2\pi} \frac{d\phi}{2\pi} \vec{\nabla}_v \cdot \left(\vec{E}_1 + \vec{v}\times\vec{B}_1\right) f_1 \right\rangle \\ &= -\frac{q}{m}\sum_{modes}\frac{1}{V}\int_0^{2\pi}\frac{d\phi}{2\pi}\vec{\nabla}_v \cdot \left(\vec{E}_k + \vec{v}\times\vec{B}_k\right) f_{-k} \end{aligned}$$

where V is the volume and we used Fourier formalism of

$$< A(t)B(t) >= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} d\omega A(-\omega)B(\omega)$$

$$< A(z)B(z) >= \lim_{V\to\infty}\frac{1}{V}\int_{-\infty}^{\infty} d^3k(-k)B(k)$$

$$f_{-k} = f(\omega_{-k}, -\vec{k}, \vec{v}) = f^*(\omega_k, \vec{k}, \vec{v}) \;\; (\because f_1(\vec{r}, \vec{v}, t) \text{ is real.})$$

since $\vec{B_1} = \frac{\vec{k}}{\omega} \times \vec{E_1}$

$$\vec{E_k} + \vec{v} \times \vec{B_k} = [1(1 - \frac{\vec{k} \cdot \vec{v}}{\omega}) + \frac{\vec{k}\vec{v}}{\omega}] \cdot \vec{E_k}$$

Also,

$$\begin{aligned}
\vec{v} &= \hat{x} v_\perp \cos\phi + \hat{y} v_\perp \sin\phi + \hat{z} v_\parallel = \hat{\rho} v_\perp + \hat{z} v_\parallel \\
\vec{\nabla}_v &= \hat{\rho} \frac{\partial}{\partial v_\perp} + \hat{\phi} \frac{1}{v_\perp} \frac{\partial}{\partial v_\phi} + \hat{z} \frac{\partial}{\partial v_\parallel} \\
\vec{k} &= \hat{x} k_\perp \cos\theta + \hat{y} k_\perp \sin\theta + \hat{z} k_\parallel \\
&= \hat{\rho} k_\perp \cos(\phi - \theta) - \hat{\phi} k_\perp \sin(\phi - \theta) + \hat{z} k_\parallel \\
\vec{E} &= \hat{x} E_x + \hat{y} E_y + \hat{z} E_z \\
&= \hat{\rho}(-E_x \cos\phi + E_y \cos\phi) + \hat{z} E_z
\end{aligned}$$

where $\hat{\rho}$ is the unit vector in the direction of $\vec{v_\perp}$, and $\hat{\phi} = \hat{z} \times \hat{\rho}$.

$$
\begin{aligned}
\vec{\nabla}_v \cdot (\vec{E} + \vec{v} \times \vec{B})_k f_{-k} &= \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp [(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\rho + \frac{1}{v_\perp}\frac{\partial}{\partial v_\phi} v_\perp [(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\phi \\
&+ \frac{\partial}{\partial v_\parallel} v_\perp [(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\parallel \\
(1)[(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\rho &= [(1(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}) + \frac{\vec{k}\vec{v}}{\omega}) \cdot \vec{E}_k]_\rho f_{-k} \\
&= [1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)](E_{kx}\cos\phi + E_{ky}\sin\phi) f_{-k} \\
&+ \frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_z] k_{perp}\cos(\phi - \theta) f_{-k} \\
\therefore \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} v_\perp [\cdots]_\rho &= \frac{1}{v_\perp}([\cdots]_\rho) + \frac{\partial}{\partial v_\perp}[\cdots]_\rho \\
&= \frac{1}{v_\perp}[\cdots]_\rho + [-\frac{k_\perp}{\omega}\cos(\phi - \theta)(E_{kx}\cos\phi + E_{ky}\sin\phi) f_{-k}] \\
&+ [1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)](E_{kx}\cos\phi + E_{ky}\sin\phi)\frac{\partial f_{-k}}{\partial v_\perp} \\
&+ \frac{1}{\omega}(E_{kx}\cos\phi + E_{ky}\sin\phi) k_\perp \cos(\phi - \theta) f_{-k} \\
&+ \frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_z] k_\perp \cos(\phi - \theta)\frac{\partial f_{-k}}{\partial v_\perp} \\
(2)[(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\phi &= [(1(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}) + \frac{\vec{k}\vec{v}}{\omega}) \cdot \vec{E}_k]_\phi f_{-k} \\
&= [1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)](-E_{kx}\sin\phi + E_{ky}\cos\phi) f_{-k} \\
&+ \frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_z](-k_\perp \sin(\phi - \theta)) f_{-k} \\
\therefore \frac{1}{v_\perp}\frac{\partial}{\partial v_\phi}[\cdots]_\phi &= \frac{1}{v_\perp}[1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)](-E_{kx}\sin\phi + E_{ky}\cos\phi)\frac{\partial f_{-k}}{\partial v_\phi} \\
&+ \frac{1}{v_\perp}\frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_z](-k_\perp \sin(\phi - \theta))\frac{\partial f_{-k}}{\partial v_\phi} \\
(3)[(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}]_\parallel &= [(1(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}) + \frac{\vec{k}\vec{v}}{\omega}) \cdot \vec{E}_k]_\parallel f_{-k} \\
&= [1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)] E_{kz} f_{-k} \\
&+ \frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_{kz}] k_\parallel f_{-k} \\
\therefore \frac{\partial}{\partial v_\parallel}[\cdots]_\parallel &= k_\parallel E_{kz} f_{-k} + [1 - \frac{1}{\omega}(k_\perp v_\perp \cos(\phi - \theta) + k_\parallel v_\parallel)] E_{kz}\frac{\partial f_{-k}}{\partial v_\parallel} \\
&+ \frac{1}{\omega} E_{kz} k_\parallel f_{-k} + \frac{1}{\omega}[v_\perp (E_{kx}\cos\phi + E_{ky}\sin\phi) + v_\parallel E_{kz}] k_\parallel \frac{\partial f_{-k}}{\partial v_\phi}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\vec{\nabla}_v \cdot [(\vec{E} + \vec{v} \times \vec{B})_k f_{-k}] &= \cos(\phi - \theta)[({E_k}^+ + {E_k}^-) S f_{-k} - E_{kz} T f_{-k}] \\
&- i \sin(\phi - \theta)({E_k}^+ + {E_k}^-) S f_{-k} + E_{kz}\frac{\partial f_{-k}}{\partial v_\parallel} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\phi}(\cdots)
\end{aligned}
$$

where

$$Sf_{-k} = (1 - \frac{k_\| v_\|}{\omega})\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}(v_\perp f_{-k}) + \frac{k_\| v_\perp}{\omega}\frac{\partial f_{-k}}{\partial v_\|}$$
$$Tf_{-k} = \frac{k_\perp v_\perp}{\omega}\frac{\partial f_{-k}}{\partial v_\|} - \frac{k_\perp v_\|}{\omega}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}(v_\perp f_{-k})$$

$$E_k^\pm = \frac{1}{2}(E_{kx} \pm iE_{ky})e^{\mp i\theta}$$

and

$$E_{kx} \equiv E_x(\vec{k}),\ E_{ky} \equiv E_y(\vec{k}),\ E_{kz} \equiv E_z(\vec{k})$$

(※ $E_{kx}\cos\phi + E_{ky}\sin\phi = \cos(\phi-\theta)(E_k^+ + E_k^-) - i\sin(\phi-\theta)(E_k^+ - E_k^-)$

Ans.)

$$\begin{aligned} E_k^+ + E_k^- &= \frac{1}{2}[E_{kx}e^{-i\theta} + iE_{ky}e^{-i\theta}] + \frac{1}{2}[E_{kx}e^{i\theta} - iE_{ky}e^{i\theta}] \\ &= \frac{1}{2}[E_{kx}(\cos\theta - i\sin\theta) + iE_{ky}(\cos\theta - i\sin\theta)] \\ &+ \frac{1}{2}[E_{kx}(\cos\theta + i\sin\theta) - iE_{ky}(\cos\theta + i\sin\theta)] \\ &= E_{kx}\cos\theta + E_{ky}\sin\theta \end{aligned}$$

$$\begin{aligned} E_k^+ - E_k^- &= -iE_{kx}\sin\theta + iE_{ky}\cos\theta \\ &= -i(E_{kx}\sin\theta - E_{ky}\cos\theta) \end{aligned}$$

$$\begin{aligned} \cos(\phi-\theta)(E_k^+ + E_k^-) &- i\sin(\phi-\theta)(E_k^+ - E_k^-) \\ &= (\cos\phi\cos\theta + \sin\phi\sin\theta)(E_{kx}\cos\theta + E_{ky}\sin\theta) \\ &\quad -i(\sin\phi\cos\theta - \cos\phi\sin\theta)(-i)(E_{kx}\sin\theta - E_{ky}\cos\theta) \\ &= E_{kx}\cos\phi + E_{ky}\sin\phi) \end{aligned}$$

But,

$$\begin{aligned} f_k &= -\frac{q}{m}\sum_n\sum_m e^{-i(n-m)(\phi-\theta)}J_m(\frac{k_\perp v_\perp}{\Omega})J_n(\frac{k_\perp v_\perp}{\Omega})\int_0^\infty d\tau \exp\left[(\omega - k_\| v_\| - n\Omega)\tau\right] \\ &\quad \times\{\cos(\phi - \theta + \Omega\tau)[(E_k^+ + E_k^-)U' - E_{kz}V] \\ &\quad -i\sin(\phi - \theta + \Omega\tau)(E_k^+ - E_k^-)U' + E_{kz}\frac{\partial f_0}{\partial v_\|}\} \end{aligned}$$

Remember f_1 obtained for the calculation of the first-order Vlasov equation with the simple replacement of ϕ by $\phi - \theta$ and τ by $-\tau$
where

$$\begin{aligned} U' &= \frac{\partial f_0}{\partial v_\perp} + \frac{k_\|}{\omega}(v_\perp\frac{\partial f_0}{\partial v_\|} - v_\|\frac{\partial f_0}{\partial v_\perp}) = \frac{1}{\omega}U \\ V &= \frac{k_\perp}{\omega}(v_\perp\frac{\partial f_0}{\partial v_\|} - v_\|\frac{\partial f_0}{\partial v_\perp}) \\ W' &= (1 - \frac{n\Omega}{\omega})\frac{\partial f_0}{\partial v_\|} + \frac{n\Omega}{\omega v_\perp}v_z\frac{\partial f_0}{\partial v_\|} = \frac{1}{\omega}W \end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \vec{\nabla}_v \cdot [(\vec{E}_k + \vec{v}\times\vec{B}_k)^* f_k]$$

$$\begin{aligned}
= \sum_{n=-\infty}^{\infty} \{[(E_k^+ + E_k^-)S - E_{kz}T]\frac{n}{\lambda}J_n(\lambda) + (E_k^+ - E_k^-)SJ_n'(\lambda) \\
+E_{kz}J_n(\lambda)\frac{\partial}{\partial v_\parallel}\}^*\{(-\frac{q}{m}\int_0^\infty d\tau\ e^{i(\omega-k_\parallel v_\parallel - n\Omega)\tau})\} \\
\times\{\frac{n}{\lambda}J_n(\lambda)[(E_k^+ + E_k^-)U' - E_{kz}V] \\
+J_n'(\lambda)(E_k^+ - E_k^-)U' + J_n(\lambda)E_{kz}\frac{\partial f_0}{\partial v_\parallel}\}
\end{aligned}$$

※ ϕ integral $\sum_n \sum_m \to \sum_n$, $\int d\tau \to \frac{n}{\lambda}J_n, J_n'$
But,

$$\begin{aligned}
-\frac{n}{\lambda}V + \frac{\partial f_0}{\partial v_\parallel} &= -\frac{n\Omega}{k_\perp v_\perp}(\frac{k_\perp v_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel} - \frac{k_\perp v_\parallel}{\omega}\frac{\partial f_0}{\partial v_\perp}) + \frac{\partial f_0}{\partial v_\parallel} \\
&= (1-\frac{n\Omega}{\omega})\frac{\partial f_0}{\partial v_\parallel} + \frac{n\Omega v_\parallel}{\omega v_\perp}\frac{\partial f_0}{\partial v_\perp} = W' \\
&= \frac{v_\parallel}{v_\perp}U' + \frac{\omega - k_\parallel v_\parallel - n\Omega}{\omega}(\frac{\partial f_0}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{\partial f}{\partial v_\perp}) \\
&= \frac{v_\parallel}{v_\perp}U'
\end{aligned}$$

The Coefficient of $E_{kz}^* J_n(\lambda)$

$$\begin{aligned}
-\frac{n}{\lambda}T + \frac{\partial}{\partial v_\parallel} &= -\frac{n\Omega}{k_\perp v_\perp}(\frac{k_\perp v_\perp}{\omega}\frac{\partial}{\partial v_\parallel} - \frac{k_\perp v_\parallel}{\omega}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp) + \frac{\partial}{\partial v_\parallel} \\
&= (1-\frac{n\Omega}{\omega})\frac{\partial}{\partial v_\parallel} + \frac{n\Omega v_\parallel}{\omega v_\perp}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp \\
&= \frac{v_\parallel}{v_\perp}S + \frac{\omega - k_\parallel v_\parallel - n\Omega}{\omega}(\frac{\partial}{\partial v_\parallel} - \frac{v_\parallel}{v_\perp}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp) \\
&= \frac{v_\parallel}{v_\perp}S
\end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \vec{\nabla}_v \cdot \left[\left(\vec{E}_k + \vec{\omega}\times\vec{B}_k\right)^* f_k\right]$$

$$\longrightarrow -i\frac{q}{m}\sum SA_k^* \frac{1}{\omega - k_\parallel v_\parallel - n\Omega}A_k U'$$

$$\begin{aligned}
\textit{where}\ \ A_k &= v_\perp\left\{E_k^+\left[\frac{n}{\lambda}J_n(\lambda) + J_n'(\lambda)\right] + E_k^-\left[\frac{n}{\lambda}J_n(\lambda) - J_n'(\lambda)\right] + v_\parallel E_{kz}J_n(\lambda)\right\} \\
&= v_\perp E_k^+ J_{n-1}(\lambda) + v_\perp E_k^- J_{n+1}(\lambda) + v_\parallel E_{kz}J_n(\lambda)
\end{aligned}$$

Thus, a remarkably compact expression for "quasi-linear evolution"

$$\frac{\partial f_0(v_\perp, v_\parallel, t)}{\partial t} = \frac{\pi q^2}{m^2} \sum_{modes} \frac{1}{\omega_k^2} \sum_{-\infty}^{\infty} L\delta(\omega_k - k_\parallel v_\parallel - n\Omega)|A|^2 L f_0$$

L is the operator, such that

$$L = (\omega_k - k_\parallel v_\parallel)\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} = n\Omega \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel}$$

We used Plemelj relation

$$\frac{1}{\omega k v} = \underline{P}\left(\frac{1}{\omega k v}\right) - i\pi\delta(\omega - kv)$$

$$J_{n-1}(x) = \frac{n}{x}J_n(x) + J_n'(x)$$
$$J_{n+1}(x) = \frac{n}{x}J_n(x) - J_n'(x)$$

Thus,

$$\frac{\partial f_0(\vec{v}, t)}{\partial t} = \pi\left(\frac{e}{m\omega}\right)^2 \sum_{-\infty}^{\infty}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[n\Omega\delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2\left(\frac{n\Omega}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel}\right)\right]\right.$$
$$\left. + \frac{\partial}{\partial v_\parallel}\left[k_\parallel \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2\left(\frac{n\Omega}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel}\right)\right]\right\}$$

$$Let \quad E^+ = E_x + iE_y \text{ (left-hand polarization)}$$
$$E^- = E_x - iE_y \text{ (right-hand polarization)}$$

$$A_k = \frac{1}{2}\left(v_\perp E^+ e^{-i\theta} J_{n-1} + v_\perp E^- e^{i\theta} J_{n+1} + 2\frac{v_\parallel}{v_\perp} E_z J_n\right) \Rightarrow \ \frac{1}{2}A$$

$$\therefore \frac{\partial f_0}{\partial t} = \pi\left(\frac{e}{2m\omega}\right)^2 \sum_{-\infty}^{\infty}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[n\Omega\delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2\left(\frac{n\Omega}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel}\right)\right]\right.$$
$$\left. + \frac{\partial}{\partial v_\parallel}\left[k_\parallel \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2\left(\frac{n\Omega}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel}\right)\right]\right\}$$

where

$$A = v_\perp E^+ e^{-i\theta} J_{N-1} + v_\perp E^- e^{i\theta} J_{N+1} + 2\frac{v_\parallel}{v_\perp} E_z J_N$$