C Quasi-linear Theory

• Velocity-space diffusion rate

\[ D_v = \frac{(\nabla v)^2}{2t} \propto E^2 \]

where E is the amplitudes of the linear-theory modes

Squares of E → “Quasi-linear”

• Motivation of the development of quasi-linear theory.
  – Microinstabilities form an important part of wave theory.
  – And the questions arise, how will the mode amplitude grow?
  – What is the instability saturation mechanism?
  – Bibliography

  * A.A.Vedenov, E.P.Velikhov and R.Z. Sagdeev (1961): “Non-linear Oscillations of Rare field Plasma” Nuclear Fusion 1, 82 (1961)

  – In which it was found that a temporally growing micro-instability acts back on the zero-order velocity distribution function.
  – Its effect on the distribution function is to produce velocity-space diffusion.
  – The diffusion tends to flatten \( f_0(\vec{v}) \) in this region and drive the instability growth rate to zero.
  – This saturation process can be viewed as a continuous diffusion through which the zero-order distribution function evolves slowly in time.
  – Two assumptions in quasi-linear theory.

  * The amplitudes of the perturbations in the plasma are not so large as to invalidate the use of zero-order orbits and of the spatially averaged distribution function, \( f_0(\vec{r}, \vec{v}, t) = \langle f_0(\vec{r}, \vec{v}, t) \rangle \)

  * The effective wave spectrum should be sufficiently dense. Any appreciable coherence between modes will be destroyed by phase mixing.

• Electromagnetic Quasi-linear Theory

\[ \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q}{m} \nabla \cdot (\vec{E} + \vec{v} \times \vec{B}) f = 0 \quad (* f = f(\vec{r}, \vec{v}, t)) \]

Averaging over a number of space and time periods of the rapid fluctuations. In addition, in presence of \( \vec{B}_0 \), we also average over the
Figure 38: Velocity distribution for “bump-on-tail” instability. Real part of unstable frequencies are such that $v = \omega_0(k)/k$ lies in region where $vdf_0/dv$ is positive (opposite sense to Landau damping). Quasi-linear diffusion due to these modes tends to flatten out the bump.

Gyro-angle in velocity space:

$$\frac{\partial f_0(v, t)}{\partial t} = \left\langle \frac{\partial f}{\partial t} \right\rangle = -\left\langle \frac{\partial}{\partial z} \frac{\partial f}{\partial z} \right\rangle - \frac{q}{m} \left\langle \nabla_v \cdot \left( \vec{E} + \vec{v} \times \vec{B} \right) f \right\rangle$$

Slow evolution of $f_0(v, t) = \left\langle f(\vec{r}, \vec{v}, t) \right\rangle$

$$\left\{ \begin{array}{l}
\text{Space averaging : } \left\langle \frac{\partial f}{\partial t} \right\rangle = 0 \\
< E >= 0 \\
\text{Higher order contributions to } \left\langle \nabla_v \cdot \left( \vec{E} + \vec{v} \times \vec{B} \right) f \right\rangle \text{ are neglected}
\end{array} \right.$$
\[ f_{-\vec{k}} = f(\omega_{-\vec{k}}, -\vec{k}, \vec{u}) = f^*(\omega_{\vec{k}}, \vec{k}, \vec{v}) \quad (\because f_1(\vec{r}, \vec{v}, t) \text{ is real.}) \]

since \( \vec{B}_1 = \frac{\vec{k}}{\omega} \times \vec{E}_1 \)

\[
\vec{E}_k + \vec{v} \times \vec{B}_k = |1(1 - \frac{\vec{k} \cdot \vec{v}}{\omega}) + \frac{\vec{k} \vec{v}^2}{\omega}| \cdot \vec{E}_k
\]

Also,

\[
\vec{v} = \hat{x}v_\perp \cos \phi + \hat{y}v_\perp \sin \phi + \hat{z}v_\parallel = \hat{\rho}v_\perp + \hat{z}v_\parallel
\]

\[
\vec{v} = \hat{x}v_\perp \cos \phi + \hat{y}v_\perp \sin \phi + \hat{z}v_\parallel = \hat{\rho}v_\perp + \hat{z}v_\parallel
\]

\[
\vec{k} = \hat{x}k_\perp \cos \theta + \hat{y}k_\perp \sin \theta + \hat{z}k_\parallel
\]

\[
\vec{k} = \hat{x}k_\perp \cos \theta + \hat{y}k_\perp \sin \theta + \hat{z}k_\parallel
\]

\[
\vec{E}_1 = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z
\]

\[
\vec{E}_1 = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z
\]

where \( \hat{\rho} \) is the unit vector in the direction of \( v_\perp^* \), and \( \hat{\phi} = \hat{z} \times \hat{\rho} \).
\[ \vec{\nabla}_v \cdot (\vec{E} + \vec{v} \times \vec{B})_{k-f-k} = \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} v_{\perp}[(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_\rho + \frac{1}{v_{\perp}} \frac{\partial}{\partial v_\phi} v_{\perp}[(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_\phi \\
+ \frac{\partial}{\partial v_{\parallel}} v_{\perp}[(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_{\parallel} \]

\[ (1) [(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_\rho = [(1(1 - \frac{k \cdot \vec{v}}{\omega}) + \frac{k_{\perp} \vec{v}}{\omega}) \cdot \vec{E}_k]_\rho f_{-k} \]

\[ = [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] (E_{kz} \cos \phi + E_{k_y} \sin \phi) f_{-k} \]

\[ + \frac{1}{\omega} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] k_{\perp} \sin(\phi - \theta) f_{-k} \]

\[ \therefore \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} v_{\perp}[(\cdot)]_\rho = \frac{1}{v_{\perp}} [(\cdot)]_\rho + \frac{\partial}{\partial v_{\perp}} [(\cdot)]_\rho \]

\[ = \frac{1}{v_{\perp}} [(\cdot)]_\rho - \frac{k_{\perp} \cos(\phi - \theta)(E_{kx} \cos \phi + E_{k_y} \sin \phi) f_{-k}] \]

\[ + [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] (E_{kz} \cos \phi + E_{k_y} \sin \phi) \frac{\partial f_{-k}}{\partial v_{\perp}} \]

\[ + \frac{1}{\omega} (E_{kx} \cos \phi + E_{k_y} \sin \phi) k_{\parallel} \sin(\phi - \theta) f_{-k} \]

\[ + \frac{1}{\omega} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] k_{\perp} \cos(\phi - \theta) \frac{\partial f_{-k}}{\partial v_{\perp}} \]

\[ (2) [(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_\phi = [(1(1 - \frac{k \cdot \vec{v}}{\omega}) + \frac{k_{\perp} \vec{v}}{\omega}) \cdot \vec{E}_k]_\phi f_{-k} \]

\[ = [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] (-E_{kz} \sin \phi + E_{k_y} \cos \phi) f_{-k} \]

\[ + \frac{1}{\omega} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] (-k_{\perp} \sin(\phi - \theta)) f_{-k} \]

\[ \therefore \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\phi}} [(\cdot)]_\phi = \frac{1}{v_{\perp}} [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] (-E_{kx} \sin \phi + E_{k_y} \cos \phi) \frac{\partial f_{-k}}{\partial v_{\phi}} \]

\[ + \frac{1}{\omega} \frac{1}{v_{\perp}} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] (-k_{\perp} \sin(\phi - \theta)) \frac{\partial f_{-k}}{\partial v_{\phi}} \]

\[ (3) [(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}]_{\parallel} = [(1(1 - \frac{k \cdot \vec{v}}{\omega}) + \frac{k_{\perp} \vec{v}}{\omega}) \cdot \vec{E}_k]_{\parallel} f_{-k} \]

\[ = [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] E_{kz} f_{-k} \]

\[ + \frac{1}{\omega} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] k_{\parallel} f_{-k} \]

\[ \therefore \frac{\partial}{\partial v_{\parallel}} [(\cdot)]_{\parallel} = k_{\parallel} E_{kz} f_{-k} + [1 - \frac{1}{\omega}(k_{\perp} v_{\perp} \cos(\phi - \theta) + k_{\parallel} v_{\parallel})] E_{kz} \frac{\partial f_{-k}}{\partial v_{\parallel}} \]

\[ + \frac{1}{\omega} E_{kz} k_{\parallel} f_{-k} + \frac{1}{\omega} [v_{\perp} (E_{kx} \cos \phi + E_{k_y} \sin \phi) + v_{\parallel} E_{z}] k_{\parallel} \frac{\partial f_{-k}}{\partial v_{\phi}} \]

Thus,

\[ \vec{\nabla}_v \cdot [(\vec{E} + \vec{v} \times \vec{B})_{k-f-k}] = \cos(\phi - \theta) [(E_{k+} + E_{k-}) S f_{-k} - E_{kz} T f_{-k}] \]

\[ - i \sin(\phi - \theta) (E_{k+} + E_{k-}) S f_{-k} + E_{kz} \frac{\partial f_{-k}}{\partial v_{\parallel}} + \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\phi}} (\cdot) \]

183
where

\[ S_{f-k} = (1 - \frac{k||v||}{\omega}) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} (v_\perp f_{-k}) + \frac{k||v||}{\omega} \frac{\partial f_{-k}}{\partial v_\parallel} \]

\[ T_{f-k} = \frac{k_\perp v_\perp}{\omega} \frac{\partial f_{-k}}{\partial v_\parallel} - \frac{k_\perp v_\parallel}{\omega} \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} (v_\perp f_{-k}) \]

\[ E_k^\pm = \frac{1}{2} (E_{kx} \pm iE_{ky}) e^{\mp i\theta} \]

and

\[ E_{kx} \equiv E_\times(\vec{k}), \quad E_{ky} \equiv E_\parallel(\vec{k}), \quad E_{kz} \equiv E_\perp(\vec{k}) \]

\[ (\times) \quad E_{kx} \cos \phi + E_{ky} \sin \phi = (\cos(\phi - \theta)(E_k^+ + E_k^-) - i \sin(\phi - \theta)(E_k^+ - E_k^-) \]

Ans.)

\[ E_k^+ + E_k^- = \frac{1}{2} [E_{kx} e^{-i\theta} + iE_{ky} e^{-i\theta}] + \frac{1}{2} [E_{kx} e^{i\theta} - iE_{ky} e^{i\theta}] \]

\[ = \frac{1}{2} E_{kx} (\cos \theta - i \sin \theta) + iE_{ky} (\cos \theta - i \sin \theta)] \]

\[ E_k^+ - E_k^- = -iE_{kx} \sin \theta + iE_{ky} \cos \theta \]

\[ = -i(E_{kx} \sin \theta - E_{ky} \cos \theta) \]

\[ \cos(\phi - \theta)(E_k^+ + E_k^-) = \frac{i}{2} (\sin(\phi - \theta)(E_k^+ - E_k^-) \]

\[ = (\cos \phi \cos \theta + \sin \phi \sin \theta) (E_{kx} \cos \theta + E_{ky} \sin \theta) \]

\[ -i(\sin \phi \cos \theta - \cos \phi \sin \theta)(-i)(E_{kx} \sin \theta - E_{ky} \cos \theta) \]

\[ = E_{kx} \cos \phi + E_{ky} \sin \phi \]

But,

\[ f_k = -\frac{q}{m} \sum_n \sum_m e^{-i(n-m)(\phi-\theta)} J_m(\frac{k_\parallel v_\parallel}{\Omega}) J_n(\frac{k_\perp v_\perp}{\Omega}) \int_0^\infty d\tau \exp \left[ i(\omega - k_\parallel v_\parallel - n\Omega)\tau \right] \]

\[ \times \{ \cos(\phi - \theta + \Omega\tau)(E_k^+ + E_k^-)U' - E_{kz} V \] \]

\[ -i \sin(\phi - \theta + \Omega\tau)(E_k^+ - E_k^-)U' + E_{kz} \frac{\partial f_0}{\partial v_\parallel} \}

Remember \( f_1 \) obtained for the calculation of the first-order Vlasov equation with the simple replacement of \( \phi \) by \( \phi - \theta \) and \( \tau \) by \( -\tau \)

where

\[ U' = \frac{\partial f_0}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega} \frac{\partial f_0}{\partial v_\parallel} - v_\perp \frac{\partial f_0}{\partial v_\perp} = \frac{1}{\omega} U \]

\[ V = \frac{k_\perp}{\omega} (v_\perp \frac{\partial f_0}{\partial v_\parallel} - v_\parallel \frac{\partial f_0}{\partial v_\perp}) \]

\[ W' = \left( 1 - \frac{n\Omega}{\omega} \right) \frac{\partial f_0}{\partial v_\parallel} + \frac{n\Omega}{\omega v_\parallel} v_\perp \frac{\partial f_0}{\partial v_\perp} = \frac{1}{\omega} W \]
\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi \nabla_v \cdot [(E_k + \vec{v} \times \vec{B}_k)^* f_k]
\]

\[
= \sum_{n=-\infty}^{\infty} \{[(E_k^+ + E_k^-)S - E_{kz} T] \frac{n}{\lambda} J_n(\lambda) + (E_k^+ - E_k^-) SJ_n'(\lambda)
\]

\[
+ E_{kz} J_n(\lambda) \frac{\partial}{\partial v} \} \{(- \frac{q}{m} \int_0^{\infty} d\tau \ e^{i(\omega - k||\vec{v}|| - n\Omega)\tau})
\]

\[
\times \{ \frac{n}{\lambda} J_n(\lambda) [(E_k^+ + E_k^-)U' - E_{kz} V]
\]

\[
+ J_n'(\lambda)(E_k^+ - E_k^-)U' + J_n(\lambda) E_{kz} \frac{\partial f_0}{\partial v} \}
\]

\[
\phi \text{ integral } \sum_n \sum_m \rightarrow \sum_n, \int d\tau \rightarrow \frac{n}{\lambda} J_n, J_n'
\]

But,

\[
- \frac{n}{\lambda} T + \frac{\partial}{\partial v} f_0 = - \frac{n\Omega}{k || v ||} \left( \frac{k || v \|}{\omega} \frac{\partial}{\partial v} v || - \frac{k \perp v \perp}{\omega} \frac{\partial}{\partial v} v \perp \right) + \frac{\partial f_0}{\partial v} ||
\]

\[
= \frac{1 - \frac{n\Omega}{\lambda}}{\omega} \frac{\partial f_0}{\partial v} || + \frac{n\Omega v \|}{\omega v \perp} \frac{\partial}{\partial v \perp} = W'
\]

\[
= \frac{v \|}{v \perp} U' + \frac{\omega - k \| v \| - n\Omega}{\omega} \left( \frac{\partial f_0}{\partial v} || - \frac{v \parallel}{v \perp} \frac{\partial f}{\partial v \perp} \right)
\]

\[
= \frac{v \||}{v \perp} U'
\]

The Coefficient of \( E_{kz} J_n(\lambda) \)

\[
- \frac{n}{\lambda} T + \frac{\partial}{\partial v} f_0 = - \frac{n\Omega}{k \| v \|} \left( \frac{k \| v \|}{\omega} \frac{\partial}{\partial v} v \| - \frac{k \perp v \perp}{\omega} \frac{1}{v \perp} \frac{\partial}{\partial v \perp} v \perp \right) + \frac{\partial}{\partial v} ||
\]

\[
= \frac{1 - \frac{n\Omega}{\lambda}}{\omega} \frac{\partial f_0}{\partial v} || + \frac{n\Omega v \|}{\omega v \perp} \frac{1}{v \perp} \frac{\partial}{\partial v \perp} v \perp
\]

\[
= \frac{v \|}{v \perp} S + \frac{\omega - k \| v \| - n\Omega}{\omega} \left( \frac{\partial}{\partial v} || - \frac{v \|}{v \perp} \frac{1}{v \perp} \frac{\partial}{\partial v \perp} v \perp \right)
\]

\[
= \frac{v \||}{v \perp} S
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi \nabla_v \cdot [(E_k + \vec{\omega} \times \vec{B}_k)^* f_k]
\]

\[
\rightarrow - \frac{i q}{m} \sum S A_k^* || - \frac{1}{\omega - k \| v \| - n\Omega} A_k U'
\]

where \( A_k = v \perp \left\{ E_k^+ \left[ \frac{n}{\lambda} J_n(\lambda) + J_n'(\lambda) \right] + E_k^- \left[ \frac{n}{\lambda} J_n(\lambda) - J_n'(\lambda) \right] + v || E_{kz} J_n(\lambda) \right\} \)

\[
= v \perp E_k^+ J_{n-1}(\lambda) + v \perp E_k^- J_{n+1}(\lambda) + v || E_{kz} J_n(\lambda)
\]
Thus, a remarkably compact expression for “quasi-linear evolution”

\[
\frac{\partial f_0(v_\perp, v_\parallel, t)}{\partial t} = \frac{\pi q^2}{m^2} \sum_{\text{modes}} \frac{1}{\omega_k^2} \sum_{\infty} L\delta(\omega_k - k_\parallel v_\parallel - n\Omega) |A|^2 Lf_0
\]

L is the operator, such that

\[
L = (\omega_k - k_\parallel v_\parallel) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} = n\Omega \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel}
\]

We used Plemelj relation

\[
\frac{1}{\omega_{kv}} = P\left(\frac{1}{\omega_{kv}}\right) - i\pi \delta(\omega - kv)
\]

\[
J_{n-1}(x) = \frac{n}{x} J_n(x) + J'_n(x)
\]

\[
J_{n+1}(x) = \frac{-n}{x} J_n(x) - J'_n(x)
\]

Thus,

\[
\frac{\partial f_0(\vec{v}, t)}{\partial t} = \pi \left(\frac{e}{m\omega}\right)^2 \sum_{-\infty}^\infty \left\{ \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \left[ n\Omega \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2 \left( \frac{n\Omega}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) \right] + \frac{\partial}{\partial v_\parallel} \left[ k_\parallel \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2 \left( \frac{n\Omega}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) \right] \right\}
\]

Let \( E^+ = E_x + iE_y \) (left-hand polarization)

\( E^- = E_x - iE_y \) (right-hand polarization)

\[
A_k = \frac{1}{2} \left( v_\perp E^+ e^{-i\theta} J_{n-1} + v_\perp E^- e^{i\theta} J_{n+1} + 2 \frac{v_\parallel}{v_\perp} E_z J_n \right) \Rightarrow \frac{1}{2} A
\]

\[
\therefore \frac{\partial f_0}{\partial t} = \pi \left(\frac{e}{2m\omega}\right)^2 \sum_{-\infty}^\infty \left\{ \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} \left[ n\Omega \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2 \left( \frac{n\Omega}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) \right] + \frac{\partial}{\partial v_\parallel} \left[ k_\parallel \delta(\omega - k_\parallel v_\parallel - n\Omega)|A|^2 \left( \frac{n\Omega}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) \right] \right\}
\]

where

\[
A = v_\perp E^+ e^{-i\theta} J_{N-1} + v_\perp E^- e^{i\theta} J_{N+1} + 2 \frac{v_\parallel}{v_\perp} E_z J_N
\]